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EUCLID's
ELEMENTS
OF
GEOMETRY,

FROM THE
Latin Tranſlation of COMMANDINE.
To which is added,
A TREATISE of the *Nature* and *Arithmetic* of
LOGARITHMS; Likewiſe
Another of the ELEMENTS of *Plane* and *Spherical*
TRIGONOMETRY; With
A PREFACE, ſhewing the *Uſefulneſs* and *Excel-
lency* of this WORK.

By Doctor *JOHN KEILL*, F. R. S. and late
Profeſſor of ASTRONOMY in *Oxford*.

The *Whole* Reviſed; where deficient, Supplied; where
loſt, or corrupted, Reſtored. Alſo,

Many Faults committed by Dr. HARRIS, Mr. CASWELL,
Mr. HEYNES, and other TRIGONOMETRICAL
WRITERS, are ſhewn; and in thoſe Caſes where
They are miſtaken, here are given Solutions *Geome-
trically* True.

An Ample Account of which may be ſeen in the PREFACE.

By *SAMUEL CUNN*.

The ELEVENTH EDITION,
Carefully Reviſed and Corrected.

To which is ſubjoined an APPENDIX, containing the
Inveſtigation of thoſe Series omitted by the AUTHOR.
And the Difference between Dr. KEILL and Mr. CUNN
impartially examined and adjuſted.

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Dr. *KEILL's*

PREFACE.

3-30-34

A *YOUNG Mathematician may be ſurprized to ſee the old obſolete* Elements *of* Euclid *appear afreſh in Print; and that, too, after ſo many new Elements of Geometry as have been lately publiſhed; eſpecially thoſe who gave us the Elements of Geometry, in a new Manner, would have us believe they have detected a great many Faults in* Euclid. *Theſe acute* Philoſophers *pretend to have diſcovered, that* Euclid's *Definitions are not perſpicuous enough; that his Demonſtrations are ſcarcely evident; that his whole Elements are ill-diſpoſed; and that they have found out innumerable Falſities in them, which had lain hid to their Times.*

But, by their Leave, I make bold to affirm, that they carp at Euclid *undeſervedly: For his Definitions are diſtinct and clear, as being taken from firſt Principles, and our moſt eaſy and ſimple Conceptions; and his Demonſtrations elegant, perſpicuous, and conciſe, carrying with them ſuch Evidence, and ſo much Strength of Reaſon, that I am eaſily induced to believe, that the Obſcurity* Sciolists *ſo often accuſe* Euclid *with, is rather to be attributed to their own per-*

plexed Ideas, than to the Demonſtrations themſelves. And, however ſome may find Fault with the Diſpoſition and Order of his Elements, yet, notwithſtanding, I do not find any Method, in all the Writings of this Kind, more proper and eaſy for Learners than that of Euclid.

It is not my Buſineſs here to anſwer, ſeparately, every one of theſe Cavillers; *but it will eaſily appear to any one, moderately verſed in theſe Elements, that they rather ſhew their own Idleneſs, than any real Faults in* Euclid. *Nay, I dare venture to ſay, there is not one of theſe new Syſtems, wherein there are not more Faults, nay, groſſer Paralogiſms, than they have been able even to imagine in* Euclid.

After ſo many unſucceſsful Endeavours in the Reformation of Geometry, ſome very good Geometricians, not daring to make new Elements, have deſervedly preferred Euclid *to all others; and have accordingly made it their Buſineſs to publiſh thoſe of* Euclid. *But they, for what Reaſon I know not, have entirely omitted ſome Propoſitions, and have altered the Demonſtrations of others, for worſe. Among whom are chiefly* Tacquet *and* Dechales, *both of which have unhappily rejected ſome elegant Propoſitions in the Elements (which ought to have been retained), as imagining them trifling and uſeleſs; ſuch, for Example, as* Prop. 27, 28, *and* 29, *of the ſixth Book, and ſome others, whoſe Uſes they might not know. Farther,* *wherever they uſe Demonſtrations of their own, inſtead of* Euclid's, *in thoſe Demonſtrations, they are faulty in their Reaſoning, and deviate very much from the Conciſeneſs of the Antients.*

In the fifth Book, they have wholly rejected Euclid's *Demonſtrations, and have given a Definition of Proportion different from* Euclid's, *and which comprehends but one of the two Species of Proportion, taking in only commenſurable Quantities. Which great Fault, no Logician or Geometrician would ever have pardoned, had not thoſe Authors done laudable Things in their other Mathematical Writings. Indeed, this Fault of theirs is common to all Modern Writers of Elements, who all ſplit on the ſame Rock; and, to ſhew their Skill, blame* Euclid, *for what, on the contrary, he ought to be commended; I mean, the Definition of proportional Quantities, wherein he ſhews an eaſy Property of thoſe Quantities, taking in both commenſurable and incommenſurable ones, and from which all the other Properties of Proportionals do eaſily follow.*

Some Geometricians, forſooth, want a Demonſtration of this Property in Euclid; *and undertake to ſupply the Deficiency by one of their own. Here, again, they ſhew their Skill in Logic, in requiring a Demonſtration for the Definition of a Term; that Definition of* Euclid *being ſuch as determines thoſe Quantities Proportionals, which have the Conditions ſpecified in the ſaid Definition.*

finition. And why might not the Author of the Elements give what Names he thought fit to Quantities, having ſuch Requiſites? Surely he might uſe his own Liberty, and accordingly has called them Proportionals.

But it may be proper here to examine the Method whereby they endeavour to demonſtrate that Property: Which is by firſt aſſuming a certain Affection, agreeing only to one Kind of Proportionals, viz. *Commenſurables; and thence, by a long Circuit, and a perplexed Series of Concluſions, do deduce that univerſal Property of Proportionals which* Euclid *affirms; a Procedure foreign enough to the juſt Methods and Rules of Reaſoning. They would certainly have done much better, if they had firſt laid down that univerſal Property by* Euclid, *and thence have deduced that particular Property agreeing to only one Species of Proportionals. But, rejecting this Method, they have taken the Liberty of adding their Demonſtration to this Definition of the fifth Book. Those who have a mind to ſee a farther Defence of* Euclid, *may conſult the Mathematical Lectures of the learned Dr.* Barrow.

As I have happened to mention this great Geometrician, I muſt not paſs by the Elements publiſhed by him, wherein, generally, he has retained the Conſtructions and Demonſtrations of Euclid *himſelf, not having omitted ſo much as one Propoſition. Hence, his Demonſtrations become more ſtrong and nervous, his Conſtructions more neat and* *elegant, and the Genius of the antient Geometricians more conſpicuous, than is uſually found in other Books of this Kind. To this he has added ſeveral Corollaries and Scholia, which ſerve not only to ſhorten the Demonſtration of what follows, but are likewiſe of Uſe in other Matters.*

Notwithſtanding this, Barrow's *Demonſtrations are ſo very ſhort, and are involved in ſo many Notes and Symbols, that they are rendered obſcure and difficult to one not verſed in Geometry. There, many Propoſitions, which appear conſpicuous in reading* Euclid *himſelf, are made knotty, and ſcarcely intelligible to Learners, by his Algebraical Way of Demonſtration; as is, for Example,* Prop. 13. Book I. *And the Demonſtrations which he lays down in* Book II. *are ſtill more difficult:* Euclid *himſelf has done much better, in ſhewing their Evidence by the Contemplations of Figures, as in Geometry ſhould always be done. The Elements of all Sciences ought to be handled after the moſt ſimple Method, and not to be involved in Symbols, Notes, or obſcure Principles, taken elſewhere.*

As Barrow's *Elements are too ſhort, ſo are thoſe of* Clavius *too prolix, abounding in ſuperfluous Scholiums and Comments: For, in my Opinion,* Euclid *is not ſo obſcure as to want ſuch a Number of Notes, neither do I doubt, but a Learner will find* Euclid *much eaſier than any of his Commentators. As too much Brevity in Geometricial Demon-*

ſtrations begets Obſcurity, ſo too much Prolixity produces Tedioußneſs and Confuſion.

On theſe Accounts, principally, it was, that I undertook to publiſh the firſt ſix Books of Euclid, *with the* 11*th and* 12*th, according to* Commandinus's *Edition; the reſt I forbore, becauſe thoſe, firſt-mentioned, are ſufficient for underſtanding of moſt Parts of the Mathematics now ſtudied.*

Farther, for the Uſe of thoſe who are deſirous to apply the Elements of Geometry to Uſes in Life, we have added a Compendium of Plane *and* Spherical Trigonometry; *by Means whereof, Geometrical Magnitudes are meaſured, and their Dimenſion expreſſed in Numbers.*

J. KEILL.

Mr. *CUNN*'s

PREFACE,

SHEWING,

The USEFULNESS *and* EXCELLENCY *of this* WORK.

DR. *KEILL*, in his Preface, hath ſufficiently declared how much eaſier, plainer, and more elegant, the Elements of Geometry written by *Euclid* are, than thoſe written by others; and that the Elements themſelves are fitter for a Learner, than thoſe publiſhed by ſuch as have pretended to comment on, ſymbolize, or tranſpoſe, any of his Demonſtrations of ſuch Propoſitions as they intend to treat of. Then how muſt a Geometrician be amazed, when he meets with a Tract * of the 1ſt, 2d, 3d, 4th, 5th, 6th, 11th, and 12th *Books* of the Elements, in which are omitted the Demonſtrations of all the Propoſitions of that moſt noble univerſal Matheſis, the 5th; on which the 6th, 11th, and 12th, ſo much depend, that the Demonſtration of not ſo much as one Propoſition, in them, can be obtained without thoſe in the fifth!

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* *Vide* the laſt Edition of the *Engliſh* Tacquet.

The 7th, 8th, and 9th Books, treat of ſuch Properties of Numbers which are neceſſary for the Demonſtration of the 10th, which treats of Incommenſurables; and the 13th, 14th, and 15th, of the five *Platonic* Bodies. But though the Doctrine of Incommenſurables, becauſe expounded in one and the ſame Plane, as the firſt ſix Elements were, claimed, by a Right Order, to be handled before Planes interſected by Planes, or the more compounded Doctrine of Solids; and the Properties of Numbers were neceſſary to the Reaſoning about Incommenſurables; yet, becauſe only one Propoſition of theſe four Books, *viz.* the 1ſt of the 10th, is quoted in the 11th and 12th Books; and that only twice, *viz.* in the Demonſtration of the 2d and 16th of the 12th; and that 1ſt Propoſition of the 10th is ſupplied by a *Lemma* in the 12th; and becauſe the 7th, 8th, 9th, 10th, 13th, 14th, and 15th Books have not been thought (by our greateſt Maſters) neceſſary to be read by ſuch as deſign to make Natural Philoſophy their Study, or by ſuch as would apply Geometry to practical Affairs; Dr. *Keill*, in his Edition, gave us only theſe eight Books, *viz.* the firſt ſix, and the 11th and 12th.

And as he found there was wanting a Treatiſe of theſe Parts of the Elements, as they were written by *Euclid* himſelf;

he publiſhed his Edition without omitting any of *Euclid*'s Demonſtrations, except two; one of which was a ſecond Demonſtration of the 9th Propoſition of the third Book; and the other a Demonſtration of that Property of Proportionals called *Converſions* (contained in a *Corollary* to the 19th Propoſition of the fifth Book;) where, inſtead of *Euclid*'s Demonſtration, which is univerſal, moſt Authors have given us only particular ones of their own. The firſt of theſe, which was omitted, is here ſupplied: And that which was corrupted is here reſtored *.

And ſince ſeveral Perſons, to whom the Elements of Geometry are of vaſt Uſe, either are not ſo ſufficiently ſkilled in, or perhaps have not Leiſure, or are not willing to take the Trouble, to read the *Latin*; and ſince this Treatiſe was not before in *Engliſh*, nor any other which may properly be ſaid to contain the Demonſtrations laid down by *Euclid* himſelf; I do not doubt but the Publication of this Edition will be acceptable, as well as ſerviceable.

Such Errors, either typographical, or in the Schemes, which were taken Notice of in the *Latin* Edition, are corrected in this.

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* *Vide Page* 55. 107. of *Euclid*'s Works, publiſhed by Dr. *Gregory*.

As to the Trigonometrical Tract, annexed to theſe Elements, I find our Author, as well as Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and others of the Trigonometrical Writers, is miſtaken in ſome of the Solutions.

That the common Solution of the 12th *Caſe* of Oblique Spherics is falſe, I have demonſtrated, and given a true one. *See Page* 318.

In the Solution of our 9th and 10th *Caſes*, by our Authors called the 1ſt and 2d, where are given and ſought oppoſite Parts, not only the afore-mentioned Authors, but all others that I have met with, have told us, that the Solutions are ambiguous; which Doctrine is, indeed, ſometimes true, but ſometimes falſe: For ſometimes the *Quæſitum* is doubtful, and ſometimes not; and when it is not doubtful, it is ſometimes greater than 90 Degrees, and ſometimes leſs: And ſure I ſhall commit no Crime, if I affirm, that no Solution can be given without a juſt Diſtinction of theſe Varieties. For the Solution of theſe *Caſes*, ſee *Pages* 320, 321.

In the Solution of our 3d and 7th *Caſes*, in other Authors reckoned the 3d and 4th, where there are given two Sides, and an Angle oppoſite to one of them, to find the 3d Side, or the Angle oppoſite to it; all the

Writers of Trigonometry, that I have met with, who have undertaken the Solutions of theſe two, as well as the two following *Caſes*, by letting fall a Perpendicular, which is undoubtedly the ſhorteſt and beſt Method for finding either of theſe *Quæſita*, have told us, that the $\left\{\begin{matrix}\text{Sum}\\ \text{Difference}\end{matrix}\right\}$ of the Vertical Angles, or Baſes, ſhall be the ſought Angle or Side, according as the Perpendicular falls $\left\{\begin{matrix}\text{without;}\\ \text{within;}\end{matrix}\right\}$ which cannot be known, unleſs the Species of that unknown Angle, which is oppoſite to a given Side, be firſt known.

Here they leave us firſt to calculate that unknown Angle, before we ſhall know whether we are to take the Sum or the Difference of the vertical Angles or Baſes for the ſought Angle or Baſe: And in the Calculation of that Angle have left us in the Dark as to its Species; as appears by the Obſervations on the two preceding *Caſes.*

The Truth is, the *Quæſitum* here, as well as in the two former *Caſes*, is ſometimes doubtful, and ſometimes not; when doubtful, ſometimes each Anſwer is leſs than 90 Degrees, ſometimes each is greater; but ſometimes one leſs, and the other greater, as in the two laſt-mentioned *Caſes.* When it is not doubtful, the *Quæ-*

ſitum is ſometimes greater than 90 Degrees, and ſometimes leſs; all which Diſtinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, oppoſite to a given Side; or, which is the ſame Thing, the Falling of the Perpendicular within or without. For which, ſee *Pages* 323, 324.

In the Solution of our 1ſt and 5th *Caſes*, called in other Authors the 5th and 6th; where there are given two Angles, and a Side oppoſite to one of them, to find the third Angle, or the Side oppoſite to it; they have told us, that the $\left\{\begin{matrix}\text{Sum}\\ \text{Difference}\end{matrix}\right\}$ of the vertical Angles, or Baſes, according as the Perpendicular falls $\left\{\begin{matrix}\text{within,}\\ \text{without,}\end{matrix}\right\}$ ſhall be the ſought Angle or Side; and that it is known whether the Perpendicular falls within, or without, by the Affection of the given Angles.

Here they ſeem to have ſpoken as tho' the *Quæſitum* was always determined, and never ambiguous; for they have here determined whether the Perpendicular falls within or without, and thereby whether they are to take the Sum or the Difference of the vertical Angles or Baſes for the ſought Angle or Side.

But, notwithſtanding theſe imaginary Determinations, I affirm, that the *Quæſitum* here, as in the two *Caſes* laſt-mentioned, is ſometimes ambiguous, and ſometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See the Solutions of theſe two *Caſes* in *Page* 322.

The Determination of the 3d *Caſe* of Oblique Plane Triangles, ſee in *Page* 224.

SAM. CUNN.

Advertiſement.

THE Reader is now preſented with a more correct Edition of this Work, than any hitherto extant; for, not only many Typographical Errors had by Degrees crept into it, but there were many Omiſſions and Miſtakes, even in the Firſt Edition, the greater Part of which have been conſtantly adhered to, in the five ſubſequent ones. Upon the Application of the Proprietors for a Reviſion of this Work, the Reviſor was favoured, by Mr. *John Robertſon*, F.R.S. late Maſter of the Royal Mathematical School in *Chriſt*'s Hoſpital, with an interleaved Copy of the firſt Edition thereof, in which are a great Number of Additions and Corrections of Mr. *Cunn*'s own Hand-writing, deſigned (as may be ſuppoſed) to have been inſerted in a Second Edition; but probably, prevented from ſo being, either by his Death, or ſome other Accident: All theſe Alterations have been carefully made, in this Edition, and ſeveral more Errors, even in that Edition which had eſcaped Mr. *Cunn*'s Notice, and have been continued in the following Editions, are in this corrected.

After theſe Amendments had been made, in the printed Copy of the Sixth Edition, the Reviſor carefully peruſed the ſame, and rectified great Numbers of falſe References to the Plates, and ſome Errors in the Plates themſelves (for they are not the ſame with thoſe annexed to the Firſt Edition): But the moſt flagrant Typographical Errors appeared in the Algebraic Series, given in the Treatiſes on Trigonometry and Logarithms, and demonſtrated in the Appendix; for the greateſt Part of theſe were ſo badly diſpoſed, as to be unintelligible, even to thoſe who underſtand the Subject; theſe are here rendered intelligible, and the Whole now is (as the Reviſor apprehends) in ſuch a State, as the ſeveral Authors of the Work and Appendix would have choſe to have put it into, had they been alive ſo to do.

EUCLID's

ELEMENTS.

BOOK I.

DEFINITIONS.

I. *A POINT is that which hath no Parts or Magnitude.*

II. *A Line is Length, without Breadth.*

III. *The Ends (or Bounds) of a Line are Points.*

IV. *A Right Line is that which lieth evenly between its Points.*

V. *A Superficies is that which hath only Length and Breadth.*

VI. *The Bounds of a Superficies are Lines.*

VII. *A plane Superficies is that which lieth evenly between its Lines.*

VIII. *A plane Angle is the Inclination of two Lines to one another in the ſame Plane, which touch each other, but do not both lie in the ſame Right Line.*

IX. *If the Lines containing the Angle be Right ones, then the Angle is called a Right-lined Angle.*

X. *When one Right Line, ſtanding on another Right Line, makes Angles on either Side there-*

of, equal between themſelves, each of theſe equal Angles is a Right one; and that Right Line, which ſtands upon the other, is called a Perpendicular to that whereon it ſtands.

XI. *An Obtuſe Angle is that which is greater than a Right one.*

XII. *An Acute Angle is that which is leſs than a Right one.*

XIII. *A Term (or Bound) is that which is the Extreme of any Thing.*

XIV. *A Figure is that which is contained under one or more Terms.*

XV. *A Circle is a plain Figure, contained under one Line, called the Circumference; to which all Right Lines, drawn from a certain Point within the Figure, are equal.*

XVI. *And that Point is called the Centre of the Circle.*

XVII. *A Diameter of a Circle is a Right Line drawn through the Centre, and terminated on both Sides by the Circumference, and divides the Circle into two equal Parts.*

XVIII. *A Semicircle is a Figure contained under a Diameter, and that Part of the Circumference of a Circle cut off by that Diameter.*

XIX. *A Segment of a Circle is a Figure contained under a Right Line, and Part of the Circumference of the Circle [which is cut off by that Right Line.]*

XX. *Right-lined Figures are ſuch as are contained under Right Lines.*

XXI. *Three-ſided Figures are ſuch as are contained under three Lines.*

XXII. *Four-ſided Figures are ſuch as are contained under four Lines.*

XXIII. *Many-ſided Figures are thoſe that are contained under more than four Right Lines.*

XXIV. *Of three-ſided Figures that is an Equilateral Triangle, which hath three equal Sides.*

XXV. *That an Iſoſceles, or Equicrural one, which hath only two Sides equal.*

XXVI. *And a Scalene one, is that, which hath three unequal Sides.*

XXVII. *Alſo of three-ſided Figures, that is a Right-angled Triangle, which hath a Right Angle.*

XXVIII. *That an Obtuſe-angled one, which hath an Obtuſe Angle.*

XXIX. *And that an Acute-angled one, which hath three Acute Angles.*

XXX. *Of four-ſided Figures, that is a Square, whoſe four Sides are equal, and its Angles all Right ones.*

XXXI. *That an Oblong, or Rectangle, which is longer than broad; but its oppoſite Sides are equal, and all its Angles Right ones.*

XXXII. *That a Rhombus, which hath four equal Sides, but not Right Angles.*

XXXIII. *That a Rhomboides, whoſe oppoſite Sides and Angles only are equal.*

XXXIV. *All Quadrilateral Figures, beſides theſe, are called Trapezia.*

XXXV. *Parallels are ſuch Right Lines, in the ſame Plane, which, if infinitely produc'd both Ways, would never meet.*

POSTULATES.

I. *GRANT that a Right Line may be drawn from any one Point to another.*

II. *That a finite Right Line may be continued directly forwards.*

III. *And that a Circle may be deſcribed about any Centre with any Diſtance.*

AXIOMS.

THINGS equal to one and the ſame Thing, are equal to one another.

II. *If to equal Things are added equal Things, the Wholes will be equal.*

III. *If from equal Things equal Things be taken away, the Remainders will be equal.*

IV. *If equal Things be added to unequal Things, the Wholes will be unequal.*

V. *If equal Things be taken from unequal Things, the Remainders will be unequal.*

VI. *Things which are double to one and the ſame Thing, are equal between themſelves.*

VII. *Things which are half one and the ſame Thing, are equal between themſelves.*

VIII. *Things which mutually agree together, are equal to one another.*

IX. *The Whole is greater than its Part.*

X. *Two Right Lines do not contain a Space.*

XI. *All Right Angles are equal between themſelves.*

XII. *If a Right Line, falling upon two other Right Lines, makes the inward Angles on the ſame Side thereof, both together, leſs than two Right Angles, thoſe two Right Lines, infinitely produc'd, will meet each other on that Side where the Angles are leſs than Right ones.*

Note, When there are ſeveral Angles at one Point, any one of them is expreſs'd by three Letters, of which that at the Vertex of the Angle is plac'd in the Middle. For Example; in the Figure of *Prop.* XIII. *Lib.* I. the Angle contain'd under the Right Lines A B, B C, is called the Angle A B C; and the Angle contained under the Right Lines A B, B E, is called the Angle A B E.

PRO-

PROPOSITION I.

PROBLEM.

To deſcribe an Equilateral Triangle upon a given finite Right Line.

LET A B be the given finite Right Line, upon which it is required to deſcribe an equilateral Triangle.

About the Centre A, with the Diſtance A B, de-
ſcribe the Circle B C D *; and about the Centre B, * *Poſt.* 3.
with the ſame Diſtance B A, deſcribe the Circle
A C E *; and from the Point C, where the two
Circles cut each other, draw the Right Lines C A,
C B †. † *Poſt.* 1.

Then becauſe A is the Centre of the Circle D B C,
A C ſhall be equal to A B ‡. And becauſe B is the ‡ *Def.* 15.
Centre of the Circle C A E, B C ſhall be equal to
B A: But C A hath been proved to be equal to A B;
therefore both C A and C B are each equal to A B.
But Things equal to one and the ſame Thing, are
equal between themſelves *, and conſequently A C is * *Ax.* 1.
equal to C B; therefore the three Sides C A, A B,
B C, are equal between themſelves.

And ſo, *the Triangle* B A C *is an Equilateral one, and is deſcribed upon the given finite Right Line* A B; which was to be done.

PROPOSITION II.

PROBLEM.

At a given Point to put a Right Line equal to a Right Line given.

LET the Point given be A, and the given Right Line B C; it is required to put a Right Line at the Point A, equal to the given Right Line B C.

* *Poſt.* 1. Draw the Right Line A C from the Point A to C *,
† 1 *of this.* upon it deſcribe the Equilateral Triangle D A C †;
‡ *Poſt.* 2. produce D A and D C directly forwards to E and G ‡;
about the Centre C, with the Diſtance B C, deſcribe
* *Poſt.* 3. the Circle B G H *; and about the Centre D, with
the Diſtance D G, deſcribe the Circle K G L.

Now becauſe the Point C is the Centre of the Circle
† *Def.* 15. B G H, B C will be equal to C G †; and becauſe D
is the Centre of the Circle K G L, the Whole D L
will be equal to the Whole D G, the Parts whereof
D A and D C are equal; therefore the Remainders
‡ *Ax.* 3. A L, C G, are alſo equal ‡. But it has been demonſtrated, that B C is equal to C G; wherefore both A L
and B C are each of them equal to C G. But Things
that are equal to one and the ſame Thing, are equal
* *Ax.* 1. to one another *; and therefore likewiſe A L is equal
to B C.

Whence, *the Right Line* A L *is put at the given Point* A, *equal to the given Right Line* B C; which was to be done.

PROPOSITION III.

PROBLEM.

Two unequal Right Lines being given, to cut off a Part from the greater, equal to the leſſer.

LET A B and C be the two unequal Right Lines given, the greater whereof is A B; it is required to cut off a Line from the greater A B equal to the leſſer C.

* 2 *of this.* Put * a Right Line A D at the Point A, equal to
the Line C; and about the Centre A, with the Diſ-
† *Poſt.* 3. tance A D, deſcribe a Circle D E F †.

Then becauſe A is the Centre of the Circle D E F,
A E is equal to A D; and ſo both A E and C are
each equal to A D; therefore A E is likewiſe equal
‡ *Ax.* 1. to C ‡.

And ſo, *there is cut off from* A B *the greater of two given Right Lines* A B *and* C, *a Line* A E *equal to the leſſer Line* C; which was to be done.

PROPOSITION IV.

THEOREM.

If there are two Triangles that have two Sides of the one equal to two Sides of the other, each to each, and the Angle contained by thoſe equal Sides in one Triangle equal to the Angle contained by the correſpondent Sides in the other Triangle; then the Baſe of one of the Triangles ſhall be equal to the Baſe of the other, the whole Triangle equal to the whole Triangle, and the remaining Angles of one equal to the remaining Angles of the other, each to each, which ſubtend the equal Sides.

LET the two Triangles be A B C, D E F, which have two Sides A B, A C, equal to two Sides D E, D F, each to each, that is, the Side A B equal to the Side D E, and the Side A C to D F; and the Angle B A C equal to the Angle E D F. I ſay, that the Baſe B C is equal to the Baſe E F, the Triangle A B C equal to the Triangle D E F, and the remaining Angles of the one equal to the remaining Angles of the other, each to its correſpondent, ſubtending the equal Sides; *viz.* the Angle A B C equal to the Angle D E F, and the Angle A C B equal to the Angle D F E.

For the Triangle A B C being applied to D E F, ſo as the Point A may co-incide with D, and the Right Line A B with D E, then the Point B will co-incide with the Point E, becauſe A B is equal to D E. And ſince A B co-incides with D E, the Right Line A C likewiſe will co-incide with the Right Line D F, becauſe the Angle B A C is equal to the Angle E D F. Wherefore alſo C will co-incide with F, becauſe the Right Line A C is equal to the Right Line D F. But the Point B co-incides with E, and therefore the Baſe B C co-incides with the Baſe E F. For, if the Point B co-inciding with E, and C with F, the Baſe B C does not co-incide with the Baſe E F; then two Right Lines will contain a Space, which is impoſſible *. Therefore, *the Baſe* B C *co-incides with the Baſe* E F, *and is equal thereto; and conſequently the whole Tri-*

* *Ax.* 10.

angle ABC *will co-incide with the whole Triangle*
DEF, *and will be equal thereto; and the remaining*
‡ *Ax.* 8. *Angles will co-incide with the remaining Angles* †, *and*
will be equal to them, viz. *the Angle* ABC *equal to*
the Angle DEF, *and the Angle* ACB *equal to the*
Angle DFE; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

The Angles at the Baſe of an Iſoſceles *Triangle are equal between themſelves: And, if the equal Sides be produced, the Angles under the Baſe ſhall be equal between themſelves.*

LET ABC be an *Iſoſceles* Triangle, having the
Side AB equal to the Side AC; and let the
equal Sides AB, AC, be produced directly forwards
to D and E. I ſay, the Angle ABC is equal to the
Angle ACB, and the Angle CBD equal to the Angle
BCE.

For aſſume any Point F in the Line BD, and from
* 3 *of this.* AE cut off the Line AG equal * to AF, and join
FC, GB.

Then, becauſe AF is equal to AG, and AB to AC,
the two Right Lines FA, AC, are equal to the two
Lines GA, AB, each to each, and contain the common
† 4 *of this.* Angle FAG; therefore the Baſe FC is equal †
to the Baſe GB, and the Triangle AFC equal to
the Triangle AGB, and the remaining Angles of
the one equal to the remaining Angles of the other,
each to each, ſubtending the equal Sides, *viz.* the Angle
ACF equal to the Angle ABG; and the Angle
AFC, equal to the Angle AGB. And becauſe the
Whole AF is equal to the Whole AG, and the Part
‡ *Ax.* 3. AB equal to the Part AC, the Remainder BF ‡ is
equal to the Remainder CG. But FC has been
proved to be equal to GB; therefore the two Sides
BF, FC, are equal to the two Sides CG, GB,
each to each, and the Angle BFC equal to the Angle
CGB; but they have a common Baſe BC. Therefore
alſo the Triangle BFC will be equal to the
* 4 *of this.* Triangle CGB *, and the remaining Angles of the one
equal

equal to the remaining Angles of the other, each to each, which ſubtend the equal Sides. And ſo the Angle F B C is equal to the Angle G C B; and the Angle B C F equal to the Angle C B G. Therefore, becauſe the whole Angle A B G has been proved equal to the whole Angle A C F, and the Part C B G equal to B C F, the remaining Angle A B C will be * equal to the remaining Angle A C B; but theſe are the Angles at the Baſe of the Triangle A C B. It hath likewiſe been proved, that the Angles F B C, G C B, under the Baſe, are equal; therefore, *the Angles at the Baſe of Iſoſceles Triangles are equal between themſelves; and if the equal Right Lines be produced, the Angles under the Baſe will be alſo equal between themſelves*; which was to be demonſtrated.

* *Ax.* 3.

Coroll. Hence every Equilateral Triangle is alſo Equiangular.

PROPOSITION VI.

Theorem.

If two Angles of a Triangle be equal, then the Sides ſubtending the equal Angles will be equal between themſelves.

LET A B C be a Triangle, having the Angle A B C equal to the Angle A C B. I ſay, the Side A B is likewiſe equal to the Side A C.

For if A B be not equal to A C, let one of them, as A B, be the greater, from which cut off B D equal to A C †, and join D C. Then, becauſe B D is equal to A C, and B C is common, D B, B C, will be equal to A C, C B, each to each, and the Angle D B C equal to the Angle A C B, from the Hypotheſis; therefore the Baſe D C is equal ‡ to the Baſe A B, and the Triangle D B C equal to the Triangle A C B, a Part to the Whole, which is abſurd; therefore A B is not unequal to A C, and conſequently is equal to it.

† 3 *of this.*

‡ 4 *of this.*

Therefore, *if two Angles of a Triangle be equal between themſelves, the Sides ſubtending the equal Angles are likewiſe equal between themſelves*; which was to be demonſtrated.

Coroll. Hence every Equiangular Triangle is alſo Equilateral.

PROPOSITION VII.

THEOREM.

On the ſame Right Line cannot be conſtituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt Right Lines have.

FOR, if it be poſſible, let two Right Lines A D, D B, equal to two others A C, C B, each to each, be conſtituted at different Points C and D, towards the ſame Part C D, and having the ſame Ends A and B, which the firſt Right Lines have, ſo that C A be equal to A D, having the ſame End A, which C A hath; and C B equal to D B, having the ſame End B.

Caſe 1. The Point D cannot fall in the Line A C; for inſtance at F: For then (A D that is) A F would not be equal to A C.

Caſe 2. If it be ſaid that D falls within the Triangle A B C; draw C D, and produce B D, B C, to F, and E. Now, ſince A D is affirmed to be equal to A C, the Angle A D C is equal to the Angle A C D *; and conſequently the Angle A C D is greater than F D C: Moreover E C D is greater than A C D, therefore E C D is much greater than F D C. But it is alſo ſaid, that B D is equal to B C, and ſo the Angle E C D under the Baſe of the Iſoſceles Triangle is equal to F D C *; whereas it hath been proved to be much greater, which is abſurd: Therefore D doth not fall within the Triangle.

* 5 *of this.*

Caſe 3. Suppoſe D fell without the Triangle A B C; join C D.

Then, becauſe A C is equal to A D, the Angle A C D will be equal † to the Angle A D C, and conſequently the Angle A D C is greater than the Angle B C D; wherefore the Angle B D C will be much greater than the Angle B C D. Again, becauſe C B is equal to D B, the Angle B D C will be equal to the Angle B C D; but it has been proved to be much greater, which is impoſſible. Therefore, *on the ſame Right*

† 5 *of this.*

Right Line cannot be conſtituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt Right Lines have; which was to be demonſtrated.

PROPOSITION VIII.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſes equal, then the Angles contained under the equal Sides will be equal.

LET the two Triangles be A B C, D E F, having two Sides, A B, A C, equal to two Sides D E, D F, each to each, *viz.* A B equal to D E, and A C to D F; and let the Baſe B C be equal to the Baſe E F. I ſay, the Angle B A C is equal to the Angle E D F.

For, if the Triangle A B C be applied to the Triangle D E F, ſo that the Point B may co-incide with E, and the Right Line B C with E F, then the Point C will co-incide with F, becauſe B C is equal to E F. And ſo, ſince B C co-incides with E F, B A and A C will likewiſe co-incide with E D and D F. For if the Baſe B C ſhould co-incide with E E, and at the ſame Time the Sides B A, A C, ſhould not co-incide with the Sides E D, D F, but change their Poſition, as E G, G F, then there would be conſtituted on the ſame Right Line two Right Lines, equal to two other Right Lines, each to each, at ſeveral Points, on the ſame Side, having the ſame Ends. But this is proved to be otherwiſe †; therefore it is impoſſible for the † 7 *of this.*
Sides B A, A C, not to co-incide with the Sides E D, D F, if the Baſe B C co-incides with the Baſe E F; wherefore they will co-incide, and conſequently the Angle B A C will co-incide with the Angle E D F †, and will be equal to it. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſes equal, then the Angles contained under the equal Sides will be equal*; which was to be demonſtrated.

PRO-

PROPOSITION IX.

PROBLEM.

To cut a given Right-lin'd Angle into two equal Parts.

LET B A C be a given Right-lin'd Angle, which is required to be cut into two equal Parts.

Aſſume any Point D in the Right Line A B, and
† 3 *of this.* cut off A E from the Line A C equal to A D †; join
‡ 1 *of this.* D E, and thereon make ‡ the equilateral Triangle D E F, and join A F. I ſay, the Angle B A C is cut into two equal Parts by the Line A F.

For, becauſe A D is equal to A E, and A F is common, the two Sides D A, A F, are equal to the two Sides A E, A F, and the Baſe D F is equal to the
‡ 8 *of this.* Baſe E F; therefore ‡ the Angle D A F is equal to the Angle E A F. Wherefore, *a given Right-lin'd Angle is cut into two equal Parts*; which was to be done.

PROPOSITION X.

PROBLEM.

To cut a given finite Right Line into two equal Parts.

LET A B be a given finite Right Line, required to be cut into two equal Parts.

* 1 *of this.* Upon it make * an Equilateral Triangle A B C, and
† 9 *of this.* biſect † the Angle A C B by the Right Line C D. I ſay, the Right Line A B is biſected in the Point D.

For, becauſe A C is equal to C B, and C D is common, the Right Lines A C, C D, are equal to the two Right Lines B C, C D, and the Angle A C D
‡ 4 *of this.* equal to the Angle B C D; therefore ‡ the Baſe A D is equal to the Baſe D B. And ſo, *the Right Line* A B *is biſected in the Point* D; which was to be done.

PRO-

PROPOSITION XI.

Problem.

To draw a Right Line at Right Angles to a given Right Line, from a given Point in the ſame.

LET A B be the given Right Line, and C the given Point. It is required to draw a Right Line from the Point C, at Right Angles to A B.

Aſſume any Point D in A C, and make C E equal * to C D; and upon D E make † the Equilateral Triangle F D E, and join F C. I ſay, the Right Line F C is drawn from the Point C, given in the Right Line A B, at Right Angles to A B. * 3 *of this.* † 1 *of this.*

For, becauſe D C is equal to C E, and F C is common, the two Lines D C, C F, are equal to the two Lines E C, C F; and the Baſe D F is equal to the Baſe F E. Therefore * the Angle D C F is equal to the Angle E C F; and they are adjacent Angles. But when a Right Line, ſtanding upon a Right Line, makes the adjacent Angles equal, each of the equal Angles is ‡ a Right Angle; and conſequently D C F, F C E, are both Right Angles. Therefore, *the Right Line* F C *is drawn from the Point* C *at Right Angles to* A B; which was to be done. * 8 *of this.* ‡ 10 *of this.*

PROPOSITION XII.

Problem.

To draw a Right Line perpendicular, upon a given infinite Right Line, from a Point given out of it.

LET A B be the given infinite Line, and C the Point given out of it. It is requir'd to draw a Right Line perpendicular upon the given Right Line A B, from the Point C given out of it.

Aſſume any Point D on the other Side of the Right Line A B; and about the Centre C, with the Diſtance C D, deſcribe * a Circle E D G; biſect † E G in H, and join C G, C H, C E. I ſay, there is drawn the Per- * *Poſt.* 3. † 10 *of this.*

Perpendicular CH on the given infinite Right Line AB, from the Point C given out of it.

For, becauſe GH is equal to HE, and HC is common, GH and HC are each equal to EH and HC, and the Baſe CG is equal to the Baſe CE. Therefore ‡ 8 *of this.* the Angle CHG is equal ‡ to the Angle CHE; and they are adjacent Angles. But when a Right Line, ſtanding upon another Right Line, makes the Angles equal between themſelves, each of the equal Angles * *Def.* 10. is a Right one *, and the ſaid ſtanding Right Line is call'd a Perpendicular to that which it ſtands on. Therefore, CH *is drawn perpendicular, upon a given infinite Right Line, from a given Point out of it*; which was to be done.

PROPOSITION XIII.

THEOREM.

When a Right Line, ſtanding upon a Right Line, makes Angles, theſe ſhall be either two Right Angles, or together equal to two Right Angles.

FOR let a Right Line AB, ſtanding upon the Right Line CD, make the Angles CBA, ABD. I ſay, the Angles CBA, ABD, are either two Right Angles, or both together equal to two Right Angles.

* *Def.* 10. For if CBA be equal to ABD, they are * each of † 11 *of this.* them Right Angles: But if not, draw † BE from the Point B, at Right Angles to CD. Therefore the Angles CBE, EBD, are two Right Angles: And becauſe CBE is equal to both the Angles CBA, ABE, add the Angle EBD, which is common; and ‡ *Ax.* 2. the two Angles CBE, EBD, together, are ‡ equal to the three Angles CBA, ABE, EBD, together. Again, becauſe the Angle DBA is equal to the two Angles DBE, EBA, together, add the common Angle ABC, and the two Angles DBA, ABC, are equal to the three Angles DBE, EBA, ABC, together. But it has been proved, that the two Angles CBE, EBD, together, are likewiſe equal to theſe three Angles: But Things that are equal to one and * *Ax.* 1. the ſame, are * equal between themſelves. Therefore likewiſe the Angle CBE, EBD, together, are equal

to the Angles D B A, A B C, together; but C B E, E B D, are two Right Angles. Therefore the Angles D B A, A B C, are both together equal to two Right Angles. Wherefore, *when a Right Line, ſtanding upon another Right Line, makes Angles, theſe ſhall be either two Right Angles, or together equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

If to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the ſaid two Right Lines will make but one ſtrait Line.

FOR let two Right Lines B C, B D, drawn from contrary Parts to the Point B, in any Right Line A B, make the adjacent Angles A B C, A B D, both together equal to two Right Angles. I ſay, B C, B D, make but one Right Line.

For if B D, C B, do not make one ſtrait Line, let C B and B E make one.

Then, becauſe the Right Line A B ſtands upon the Right Line C B E, the Angle A B C, A B E, together, will be equal * to two Right Angles. But the Angles A B C, A B D, together, are alſo equal to two Right Angles. Now taking away the common Angle A B C, the remaining Angle A B E is equal to the remaining Angle A B D, the leſs to the greater, which is impoſſible. Therefore B E, B C, are not one ſtrait Line. And in the ſame manner it is demonſtrated, that no other Line but B D is in a ſtrait Line with C B; wherefore C B, B D, ſhall be in one ſtrait Line. Therefore, *if to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the ſaid two Right Lines will make but one ſtrait Line*; which was to be demonſtrated.

* 13 *of this.*

PRO-

PROPOSITION XV.

Theorem.

If two Right Lines mutually cut each other, the oppoſite Angles are equal.

LET the two Right Lines A B, C D, mutually cut each other in the Point E. I ſay, the Angle A E C is equal to the Angle D E B; and the Angle C E B equal to the Angle A E D.

For, becauſe the Right Line A E, ſtanding on the Right Line C D, makes the Angles C E A, A E D; * 13 *of this.* theſe both together ſhall be equal * to two Right Angles. Again, becauſe the Right Line D E, ſtanding upon the Right Line A B, makes the Angles A E D, D E B; theſe Angles together are * equal to two Right Angles. But it has been proved, that the Angles C E A, A E D, are likewiſe together equal to two Right Angles. Therefore the Angles C E A, A E D, are equal to the Angles A E D, D E B. Take away the common Angle A E D, and the Angle remaining C E A is † *Ax.* 3. † equal to the Angle remaining B E D. For the ſame Reaſon, the Angle C E B ſhall be equal to the Angle D E A. Therefore, *if two Right Lines mutually cut each other, the oppoſite Angles are equal*, which was to be demonſtrated.

Coroll. 1. From hence it is manifeſt, that two Right Lines, mutually cutting each other, make Angles at the Section equal to four Right Angles.

Coroll. 2. All the Angles, conſtituted about the ſame Point, are equal to four Right Angles.

PROPOSITION XVI.

Theorem.

If one Side of any Triangle be produced, the outward Angle is greater than either of the inward oppoſite Angles.

LET A B C be a Triangle, and one of its Sides B C be produced to D. I ſay, the outward Angle A C D is greater than either of the inward Angles C B A, or B A C.

For biſect A C in E *, and join B E, which produce to F, and make E F equal to B E †. Moreover, join F C, and produce A C to G. * 10 of this. † 3 of this.

Then, becauſe A E is equal to E C, and B E to E F, the two Sides A E, E B, are equal to the two Sides C E, E F, each to each, and the Angle A E B † equal to the Angle F E C; for they are oppoſite Angles. Therefore the Baſe A B is ‡ equal to the Baſe F C; and the Triangle A E B equal to the Triangle F E C; and the remaining Angles of the one equal to the remaining Angles of the other, each to each, ſubtending the equal Sides. Wherefore the Angle B A E is equal to the Angle E C F; but the Angle E C D is greater than the Angle E C F; therefore the Angle A C D is greater than the Angle B A E. After the ſame manner, if the Right Line B C be biſected, we demonſtrate that the Angle B C G, and conſequently its equal, the Angle A C D *, is greater than the Angle A B C. Therefore, *one Side of any Triangle being produced, the outward Angle is greater than either of the inward oppoſite Angles*; which was to be demonſtrated. † 15 of this. ‡ 4 of this. * 15 of this.

PROPOSITION XVII.

Theorem.

Two Angles of any Triangle together, howſoever taken, are leſs than two Right Angles.

LET A B C be a Triangle. I ſay, two Angles of it together, howſoever taken, are leſs than two Right Angles.

For produce B C to D.

Then, becauſe the outward Angle A C D of the Triangle A B C is greater * than the inward oppoſite Angle A B C; if the common Angle A C B be added, the Angles A C D, A C B, together, will be greater than the Angles A B C, A C B, together: But A C D, A C B, are † equal to two Right Angles. Therefore A B C, B C A, are leſs than two Right Angles. In the ſame manner we demonſtrate, that the Angles B A C, A C B, as alſo C A B, A B C, are leſs than two Right Angles. Therefore, *two Angles of any Triangle together, howſoever taken, are leſs than two Right Angles*; which was to be demonſtrated.

* 16 of this.

† 13 of this.

PROPOSITION XVIII.

THEOREM.

The greater Side of every Triangle ſubtends the greater Angle.

LET A B C be a Triangle, having the Side A C greater than the Side A B. I ſay, the Angle A B C is greater than the Angle B C A.

For, becauſe A C is greater than A B, A D may be made equal to A B ‡, and B D be joined.

Then, becauſe A D B is an outward Angle of the Triangle B D C, it will be * greater than the inward oppoſite Angle D C B. But A D B is † equal to A B D; becauſe the Side A B is equal to the Side A D. Therefore the Angle A B D is likewiſe greater than the Angle A C B; and conſequently A B C ſhall be much greater than A C B. Wherefore, *the greater Side of every Triangle ſubtends the greater Angle*; which was to be demonſtrated.

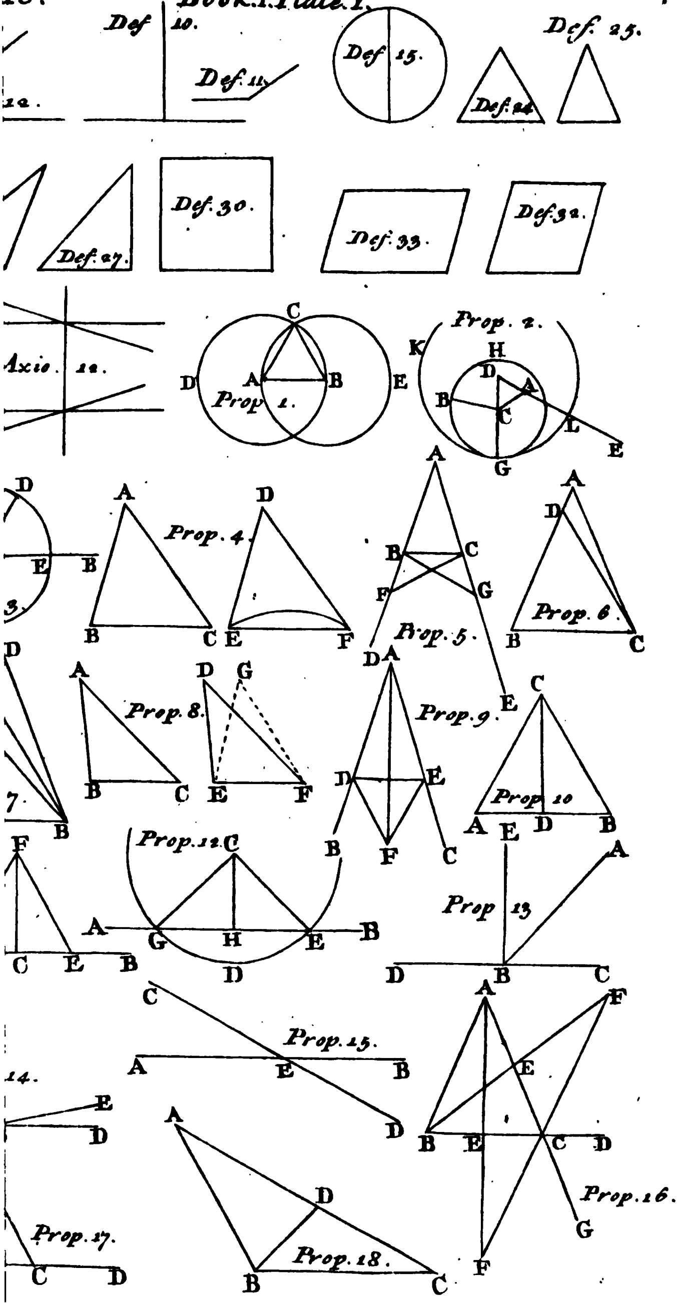
‡ 3 of this.

* 16 of this.

† 5 of this.

PRO-

18. **Book.1.Plate.1.**



OF

PROPOSITION XIX.

THEOREM.

The greater Angle of every Triangle ſubtends the greater Side.

LET A B C be a Triangle, having the Angle A B C greater than the Angle B C A. I ſay, the Side A C is greater than the Side A B.

For, if it be not greater, A C is either equal to A B, or leſs than it. It is not equal to it, becauſe then the Angle A B C would be equal * to the Angle A C B; but it is not: Therefore A C is not equal to A B. Neither will it be leſs; for then the Angle A B C would be † leſs than the Angle A C B; but it is not. Therefore A C is not leſs than A B. But likewiſe it has been proved not to be equal to it: Wherefore A C is greater than A B. Therefore, *the greater Angle of every Triangle ſubtends the greater Side*; which was to be demonſtrated.

* 5 *of this.*
† 18 *of this.*

PROPOSITION XX.

THEOREM.

Two Sides of any Triangle, howſoever taken, are together greater than the Third Side.

LET A B C be a Triangle. I ſay, two Sides thereof, howſoever taken, are together greater than the third Side; *viz.* the Sides B A, A C, are greater than the Side B C; and the Sides A B, B C, greater than the Side A C; and the Sides B C, C A, greater than the Side A B.

For produce B A to the Point D, ſo that A D be * equal to A C; and join D C.

Then, becauſe D A is equal to A C, the Angle A D C ſhall be equal † to the Angle A C D. But the Angle B C D is greater than the Angle A C D. Wherefore the Angle B C D is greater than the Angle A D C; and becauſe D C B is a Triangle, having the Angle B C D greater than the Angle B D C, and the

* 3 *of this.*
† 5 *of this.*

† 19 *of this.* greater Angle ſubtends † the greater Side; the Side DB will be greater than the Side BC. But DB is equal to BA and AC together. Wherefore the Sides BA, AC, together, are greater than the Side BC. In the ſame manner we demonſtrate, that the Sides AB, BC, together, are greater than the Side CA; and the Sides BC, CA, together, are greater than the Side AB. Therefore, *two Sides of any Triangle, howſoever taken, are together greater than the third Side*; which was to be demonſtrated.

PROPOSITION XXI.

Theorem.

If two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle.

FOR let two Right Lines BD, DC, be drawn from the Extremes B, C, of the Side BC of the Triangle ABC, to the Point D within the ſame. I ſay, BD, DC, are leſs than BA, AC, the other two Sides of the Triangle, but contain an Angle BDC greater than the Angle BAC.

For produce BD to E.

Then, becauſe two Sides of every Triangle together
* 20 *of this.* are * greater than the third, BA, AE the two Sides of the Triangle ABE, are greater than the Side BE. Now, add EC, which is common, and the Sides
† *Ax.* 4. BA, AC, will be † greater than BE, EC.

Again, becauſe CE, ED, the two Sides of the Triangle CED, are greater than the Side CD, add DB, which is common, and the Sides CE, EB, will be greater than CD, DB. But it has been proved, that BA, AC, are greater than BE, EC. Wherefore BA, AC, are much greater than BD, DC. Again,
‡ 16 *of this.* becauſe the outward Angle of every Triangle ‡ is greater than the inward and oppoſite one; BDC, the outward Angle of the Triangle CDE, ſhall be greater than the Angle CED. For the ſame Reaſon, CED, the outward Angle of the Triangle ABE, is likewiſe greater

greater than the Angle B A C; but the Angle BDC has been proved to be greater than the Angle CED. Wherefore the Angle B D C ſhall be much greater than the Angle B A C. And ſo, *if two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle*; which was to be demonſtrated.

PROPOSITION XXII.

PROBLEM.

To deſcribe a Triangle of three Right Lines, which are equal to three others given: But it is requiſite, that any two of the Right Lines taken together be greater than the third; becauſe two Sides of a Triangle, howſoever taken, are together greater than the third Side.

LET A, B, C, be three Right Lines given, two of which, any ways taken, are greater than the third; *viz.* A and B together greater than C; A and C greater than B; and B and C greater than A. Now it is required to make a Triangle of three Right Lines equal to A, B, C: Let there be one Right Line D E, terminated at D, but infinite towards E; and take* D F equal to A, F G equal to B, and G H equal to C; and about the Centre F, with the Diſtance F D, deſcribe a Circle D K L†; and about the Centre G, with the Diſtance G H, deſcribe another Circle K L H, and join K F, K G. I ſay, the Triangle K F G is made of three Right Lines, equal to A, B, C; for, becauſe the Point F is the Centre of the Circle D K L, F K ſhall be equal to F D‡: But F D is equal to A; therefore F K is alſo equal to A. Again, becauſe the Point G is the Centre of the Circle L K H, G K will be‡ equal to G H: But G H is equal to C; therefore ſhall G K be alſo equal to C: But F G is likewiſe equal to B; and conſequently the three Right Lines K F, F G, K G, are equal to the three Right Lines A, B, C. Wherefore, *the Triangle* K F G *is made of three Right Lines* K F, F G, K G, *equal to the three given Lines* A, B, C, which was to be done.

* *3 of this.*

† *3 Poſt.*

‡ *Def. 15.*

PROPOSITION XXIII.

PROBLEM.

With a given Right Line, and at a given Point in it, to make a Right-lin'd Angle equal to a Right-lin'd Angle given.

LET the given Right Line be A B, and the Point given therein A, and the given Right-lin'd Angle D C E. It is required to make a Right-lin'd Angle at the given Point A, with the given Right Line A B, equal to the given Right-lin'd Angle D C E.

Aſſume the Points D and E at Pleaſure in the Lines C D, C E, and draw D E; then, of three Right
* 22 of this. Lines equal to C D, D E, E C, make * a Triangle A F G, ſo that A F be equal to C D, A G to C E, and F G to D E.

Then, becauſe the two Sides D C, C E, are equal to the two Sides F A, A G, each to each, and the Baſe D E equal to the Baſe F G; the Angle D C E ſhall
† 8 of this. be † equal to the Angle F A G. Therefore, *the Right-lin'd Angle* F A G *is made, at the given Point* A, *in the given Right Line* A B, *equal to the given Right-lin'd Angle* D C E; which was to be done.

PROPOSITION XXIV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Angle of the one contained under the equal Right Lines, greater than the correſpondent Angle of the other; then the Baſe of the one will be greater than the Baſe of the other.

LET there be two Triangles A B C, D E F, having two Sides A B, A C, equal to the two Sides D E, D F, each to each, *viz.* the Side A B equal to the Side D E, and the Side A C equal to D F; and let the Angle B A C be greater than the Angle E D F. I ſay, the Baſe B C is greater than the Baſe E E.

For becauſe the Angle B A C is greater than the Angle E D F; make an * Angle E D G at the Point D in the Right Line D E, equal to the Angle B A C; * 23 *of this.* and make † D G equal to either A C or D F, and join E G, F G. † 3 *of this.*

Caſe 1. When E G falls above E F; then, becauſe A B is equal to D E, and A C to D G, the two Sides B A, A C, are each equal to the two Sides E D, D G, and the Angle B A C equal to the Angle E D G: Therefore the Baſe B C is equal ‡ to the Baſe E G. ‡ 4 *of this.* Again, becauſe D G is equal to D F, the Angle D F G is † equal to the Angle D G F; and ſo the Angle † 5 *of this.* D F G is greater than the Angle E G F: And conſequently the Angle E F G is much greater than the Angle E G F. And becauſe E F G is a Triangle, having the Angle E F G greater than the Angle E G F; and the greateſt Angle ſubtends * the greateſt * 19 *of this.* Side, the Side E G ſhall be greater than the Side E F. But the Side E G is equal to the Side B C: Whence B C is likewiſe greater than E F.

Caſe 2. When E G falls upon E F; then E G is greater than E F: And conſequently B C is greater than E F.

Caſe 3. When E G falls below E F; then D G, E G, are ‡ together greater than D F and E F together, ‡ 21 *of this.* and by taking away the Equals D G, D F, there remain E G greater than E F *. Therefore B C, * *Ax.* 5. which is equal to E G, will be alſo greater than E F. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Angle of the one contained under the equal Right Lines, greater than the correſpondent Angle of the other; then the Baſe of the one will be greater than the Baſe of the other*; which was to be demonſtrated.

PROPOSITION XXV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſe of the one greater than the Baſe of the other; they ſhall alſo have the Angles contained by the equal Sides, the one greater than the other.

LET there be two Triangles ABC, DEF, having two Sides AB, AC, each equal to two Sides DE, DF, *viz.* the Side AB equal to the Side DE, and the Side AC to the Side DF; but the Baſe BC greater than the Baſe EF; I ſay, the Angle BAC is alſo greater than the Angle EDF.

For if it be not greater, it will be either equal or leſs. But the Angle BAC is not equal to the Angle
* 4 *of this.* EDF; for if it was, the Baſe BC would be * equal to the Baſe EF; but it is not: Therefore the Angle BAC is not equal to the Angle EDF, neither will
† 24 *of this.* it be leſſer; for if it ſhould, the Baſe BC would be † leſs than the Baſe EF; but it is not. Therefore the Angle BAC is not leſs than the Angle EDF; but it has likewiſe been proved not to be equal to it. Wherefore the Angle BAC is neceſſarily greater than the Angle EDF. *If, therefore, two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſe of the one greater than the Baſe of the other; they ſhall alſo have the Angles contained by the equal Sides, the one greater than the other;* which was to be demonſtrated.

PROPOSITION XXVI.

THEOREM.

If two Triangles have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which ſubtends one of the equal Angles; the remaining Sides of the one Triangle ſhall be alſo equal to the remaining Sides of the other, each to his correſpondent Side; and the remaining Angle of the one equal to the remaining Angle of the other.

LET there be two Triangles A B C, D E F, having two Angles A B C, B C A, of the one, equal to two Angles D E F, E F D, of the other, each to each, that is, the Angle A B C equal to the Angle D E F, and the Angle B C A equal to the Angle E F D. And let one Side of the one be equal to one Side of the other, which firſt let be the Side lying between the equal Angles, *viz.* the Side B C equal to the Side E F. I ſay, the remaining Sides of the one Triangle will be equal to the remaining Sides of the other, each to each, that is, the Side A B equal to the Side D E, and the Side A C equal to the Side D F, and the remaining Angle B A C equal to the remaining Angle E D F.

For if the Side A B be not equal to the Side D E, one of them will be the greater, which let be A B, make G B equal to D E, and join G C.

Then, becauſe B G is equal to D E, and B C to E F, the two Sides G B, B C, are equal to the two Sides D E, E F, each to each; and the Angle G B C equal to the Angle D E F. The Baſe G C is * equal to the Baſe D F, and the Triangle G B C to the Triangle D E F, and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides. Therefore the Angle G C B is equal to the Angle D F E. But the Angle D F E, by the Hypotheſis,

* *4 of this.*

theſis, is equal to the Angle B C A; and ſo the Angle B C G is likewiſe equal to the Angle B C A, the leſs to the greater, which cannot be. Therefore A B is not unequal to D E, and conſequently is equal to it. And ſo the two Sides A B, B C, are equal to the two Sides D E, E F, and the Angle A B C equal to the Angle D E F: And conſequently the Baſe A C * is equal to the Baſe D F, and the remaining Angle B A C equal to the remaining Angle E D F.

* *4 of this.*

Secondly, Let the Sides that are ſubtended by the equal Angles be equal, as A B equal to D E. I ſay, the remaining Sides of the one Triangle are equal to the remaining Sides of the other, *viz.* A C to D F, and B C to E F; and alſo the remaining Angle B A C, to the remaining Angle E D F.

For if B C be unequal to E F, one of them is the greater, which let be B C, if poſſible, and make B H equal to E F, and join A H.

Now, becauſe B H is equal to E F, and A B to D E, the two Sides A B, B H, are equal to the two Sides D E, E F, each to each, and they contain equal Angles: Therefore the Baſe A H is † equal to the Baſe D F; and the Triangle A B H ſhall be equal to the Triangle D E F, and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides; and ſo the Angle B H A is equal to the Angle E F D. But E F D is † equal to the Angle B C A; and conſequently the Angle B H A is equal to the Angle B C A: Therefore the outward Angle B H A of the Triangle A H C, is equal to the inward and oppoſite Angle B C A; which is * impoſſible: Whence B C is not unequal to E F; therefore it is equal to it. But A B is alſo equal to D E. Wherefore the two Sides A B, B C, are equal to the two Sides D E, E F, each to each; and they contain equal Angles. And ſo the Baſe A C is equal to the Baſe D F, the Triangle B A C to the Triangle D E F, and the remaining Angle B A C equal to the remaining Angle E D F. *If,* therefore, *two Triangles have two Angles equal, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which ſubtends one of the equal Angles; the remaining Sides of the one Triangle ſhall be alſo equal to the remaining Sides of the other, each to its correſpondent*

† *From the Hyp.*

* *16 of this.*

ſpondent Side, and the remaining Angle of the one equal to the remaining Angle of the other; which was to be demonſtrated.

PROPOSITION XXVII.

Theorem.

If a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themſelves, the two Right Lines ſhall be parallel.

LET the Right Line E F, falling upon two Right Lines A B, C D, make the alternate Angles A E F, E F D, equal between themſelves. I ſay, the Right Line A B is parallel to C D.

For if it be not parallel, A B and C D, produced towards B and D, or towards A and C, will meet: Now let them be produced towards B and D, and meet in the Point G.

Then the outward Angle A E F of the Triangle
G E F is * greater than the inward and oppoſite An- * *16 of this.*
gle E F G, and alſo equal † to it; which is abſurd. † *From the*
Therefore A B and C D, produced towards B and D, *Hyp.*
will not meet each other. By the ſame Way of Rea-
ſoning, neither will they meet, being produced towards
C and A. But Lines that meet each other on neither
Side, are ‡ parallel between themſelves. Therefore ‡ *Def. 35.*
A B is parallel to C D. Therefore, *if a Right Line,*
falling upon two Right Lines, makes the alternate Angles
equal between themſelves, the two Right Lines ſhall be
parallel; which was to be demonſtrated.

PRO-

PROPOSITION XXVIII.

Theorem.

If a Right Line, falling upon two Right Lines, makes the outward Angle with the one Line equal to the outward and oppoſite Angle with the other on the ſame Side, or the inward Angles on the ſame Side together equal to two Right Angles, the two Right Lines ſhall be parallel between themſelves.

LET the Right Line EF falling upon two Right Lines AB, CD, make the outward Angle EGB equal to the inward and oppoſite Angle GHD; or the inward Angles BGH, GHD, on the ſame Side together equal to two Right Angles: I ſay, the Right Line AB is parallel to the Right Line CD.

* *From the Hyp.* For, becauſe the Angle EGB is * equal to the Angle GHD, and the Angle EGB † equal to the Angle AGH, the Angle AGH ſhall be equal to the Angle GHD; but theſe are alternate Angles. † 15 *of this.* Therefore AB is ‡ parallel to CD. ‡ 27 *of this.*

Again, becauſe the Angles BGH, GHD, are equal to two Right Angles, and AGH, BGH, are ‖ equal to two Right ones, the Angles AGH, BGH, will be equal to the Angles BGH, GHD; and if the common Angle BGH be taken from both, there will remain the Angle AGH equal to the Angle GHD; but theſe are alternate Angles. Therefore AB is parallel to CD. ‖ 13 *of this.* *If*, therefore, *a Right Line, falling upon two Right Lines, makes the outward Angle with the one Line equal to the inward and oppoſite Angle with the other on the ſame Side, or the inward Angles on the ſame Side together equal to two Right Angles, the two Right Lines ſhall be parallel between themſelves*; which was to be demonſtrated.

PROPOSITION XXIX.

THEOREM.

If a Right Line falls upon two Parallels, it will make the alternate Angles equal between themſelves; the outward Angle equal to the inward and oppoſite Angle, on the ſame Side; and the inward Angles on the ſame Side together equal to two Right Angles.

LET the Right Line E F fall upon the parallel Right Lines A B, C D. I ſay, the alternate Angles A G H, G H D, are equal between themſelves; the outward Angle E G B is equal to the inward one G H D, on the ſame Side; and the two inward ones, B G H, G H D, on the ſame Side, are together equal to two Right Angles.

For if A G H be unequal to G H D, one of them will be the greater. Let this be A G H; then becauſe the Angle A G H is greater than the Angle G H D, add the common Angle B G H to both: And ſo the Angles A G H, B G H, together, are greater than the Angles B G H, G H D, together. But the Angles A G H, B G H, are equal to two Right ones *. * 13 *of this.*
Therefore B G H, G H D, are leſs than two Right Angles. And ſo the Lines A B, C D, infinitely produced † will meet each other; but becauſe they are † *Ax.* 12.
parallel they will not meet. Therefore the Angle A G H is not unequal to the Angle G H D. Wherefore it is neceſſarily equal to it.

But the Angle A G H is ‡ equal to the Angle ‡ 15 *of this.*
E G B: Therefore E G B is alſo equal to G H D.

Now add the common Angle B G H; and then E G B, B G H, together, are equal to B G H, G H D, together; but E G B, and B G H, are equal to two Right Angles. Therefore alſo B G H, and G H D, ſhall be equal to two Right Angles. Wherefore, *if a Right Line falls upon two Parallels, it will make the alternate Angles equal between themſelves; the outward Angle equal to the inward and oppoſite Angle, on the ſame Side; and the inward Angles on the ſame Side together equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XXX.

THEOREM.

Right Lines, parallel to one and the ſame Right Line, are alſo parallel between themſelves.

LET A B and C D be Right Lines, each of which is parallel to the Right Line E F. I ſay, A B is alſo parallel to C D. For let the Right Line G K fall upon them.

Then, becauſe the Right Line G K falls upon the
* 29 *of this.* parallel Right Lines A B, E F, the Angle A G H is *
equal to the Angle G H F; and becauſe the Right
Line G K falls upon the parallel Right Lines E F, C D,
† 29 *of this.* the Angle G H F is equal to the Angle G K D †. But
it has been proved that the Angle A G K is alſo equal
to the Angle G H F. Therefore A G K is equal to
G K D, and they are alternate Angles; whence A B
† 27 *of this.* is parallel to C D †. And ſo, *Right Lines, parallel to
one and the ſame Right Line, are parallel between themſelves*; which was to be demonſtrated.

PROPOSITION XXXI.

PROBLEM.

To draw a Right Line through a given Point parallel to a given Right Line.

LET A be a Point given, and B C a Right Line given. It is required to draw a Right Line thro' the Point A, parallel to the Right Line B C.

Aſſume any Point D in B C, and join A D; then
‡ 23 *of this.* make ‡ an Angle D A E, at the Point A, with the
Line D A, equal to the Angle A D C, and produce
E A ſtrait forwards to F.

Then, becauſe the Right Line A D, falling on two
Right Lines B C, E F, makes the alternate Angles
E A D, A D C, equal between themſelves, E F ſhall
† 27 *of this.* be † parallel to B C. Therefore, *the Right Line
E A F is drawn thro' the given Point A, parallel to the
given Right Line* B C; which was to be done.

PRO-

PROPOSITION XXXII.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is equal to both the inward and oppoſite Angles; and the three inward Angles of a Triangle are equal to two Right Angles.

LET A B C be a Triangle, one of whoſe Sides B C is produced to D. I ſay, the outward Angle A C D is equal to the two inward and oppoſite Angles C A B, A B C; and the three inward Angles of the Triangle, *viz.* A B C, B C A, C A B, are equal to two Right Angles.

For let C E be drawn * thro' the Point C, parallel to the Right Line A B. Then, becauſe A B is parallel to C E, and A C falls upon them, the alternate Angles B A C, A C E, are † equal between themſelves. Again, becauſe A B is parallel to C E, and the Right Line B D falls upon them, the outward Angle E C D is ‖ equal to the inward and oppoſite one A B C; but it has been proved, that the Angle A C E is equal to the Angle B A C. Wherefore the whole outward Angle A C D is equal to both the inward and oppoſite Angles B A C, A B C. And if the Angle A C B, which is common, be added, the two Angles A C D, A C B, are equal to the three Angles A B C, B A C, A C B; but the Angles A C D, A C B, are ‡ equal to two Right Angles. Therefore alſo ſhall the Angles A C B, C B A, C A B, be equal to two Right Angles. Wherefore, *if one Side of any Triangle be produced, the outward Angle is equal to both the inward and oppoſite Angles, and the three inward Angles of a Triangle are equal to two Right Angles*; which was to be demonſtrated.

* 31 *of this.*
† 29 *of this.*
‖ 29 *of this.*
‡ 13 *of this.*

Coroll. 1. All the three Angles of any one Triangle, taken together, are equal to all the three Angles of any other Triangle, taken together.

Coroll. 2. If two Angles of any one Triangle, either ſeparately, or taken together, be equal to two Angles of any other Triangle; then the remaining Angle of the one Triangle will be equal to the remaining Angle of the other.

Coroll. 3. If one Angle of a Triangle be a Right Angle, the other two Angles together make one Right Angle.

Coroll. 4. If the Angle included between the equal Legs of an Iſoſceles Triangle be a Right one, each of the other Angles at the Baſe will be half a Right Angle.

Coroll. 5. Any Angle in an Equilateral Triangle is equal to one Third of two Right Angles, or two Thirds of one Right Angle.

Coroll. 6. Hence it appears, that if one Angle of any Triangle be equal to the other two, that is a Right one; becauſe that the Angle adjacent to this Right one, is equal to the other two. But when adjacent Angles are equal, they are neceſſarily Right ones.

THEOREM I.

All the inward Angles of any Right-lin'd Figure whatſoever, make twice as many Right Angles, abating four, as the Figure has Sides.

FOR any Right-lin'd Figure may be reſolved into as many Triangles, abating two, as it hath Sides. For Example, if a Figure has four Sides, it may be reſolved into two Triangles: If a Figure hath five Sides, it may be reſolved in three Triangles; if ſix, into four; and ſo on. Wherefore (by Prop. XXXII.) *the Angles of all theſe Triangles are equal to twice as many Right Angles as there are Triangles: But the Angles of all the Triangles are equal to the inward Angles of the Figure. Therefore all the inward Angles of the Figure are equal to twice as many Right Angles as there are Triangles, that is, twice as many Right Angles, taking away four, as the Figure hath Sides.* W. W. D.

THE-

THEOREM II.

All the outward Angles of any Right-lined Figure, together, make four Right Angles.

FOR the outward Angles, together with the inward ones, make twice as many Right Angles as the Figure has Sides; but from the laſt Theorem, all the inward Angles, together, make twice as many Right Angles, abating four, as the Figure hath Sides. Wherefore the outward Angles are, all together, equal to four Right Angles. W. W. D.

PROPOSITION XXXIII.

THEOREM.

Two Right Lines, which join two equal and parallel Right Lines, towards the ſame Parts, are alſo equal and parallel.

LET the parallel and equal Right Lines A B, C D, be joined, towards the ſame Parts, by the Right Lines A C, B D. I ſay, A C, B D, are equal and parallel.

For draw B C.

Then, becauſe A B is parallel to C D, and B C falls upon them, the alternate Angles A B C, B C D, are
* equal. Again, becauſe A B is equal to C D, and * *29 of this.*
B C is common; the two Sides A B, B C, are equal to the two Sides B C, C D; but the Angle A B C is alſo equal to the Angle B C D; therefore the Baſe A C
is † equal to the Baſe B D: And the Triangle A B C † *4 of this.*
equal to the Triangle B C D; and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides. Wherefore the Angle A C B is equal to the Angle C B D. And becauſe the Right Line B C, falling upon two Right Lines A C, B D, makes the alternate Angles A C B, C B D, equal
between themſelves; A C is ‡ parallel to B D. But it ‡ *29 of this.*
has been proved alſo to be equal to it. Therefore, *two Right Lines, which join two equal and parallel Right Lines, towards the ſame Parts, are alſo equal and parallel*; which was to be demonſtrated.

Defin. *A Parallelogram is a Quadrilateral Figure, each of whoſe oppoſite Sides are parallel.*

PROPOSITION XXXIV.

THEOREM.

The oppoſite Sides and oppoſite Angles of any Parallelogram are equal; and the Diameter divides the ſame into two equal Parts.

LET ABCD be a Parallelogram, whoſe Diameter is BC. I ſay, the oppoſite Sides and oppoſite Angles are equal between themſelves, and the Diameter B C biſects the Parallelogram.

For, becauſe A B is parallel to C D, and the Right Line B C falls on them, the alternate Angles A B C, B C D, are * equal between themſelves; again, becauſe A C is parallel to B D, and B C falls upon them, the alternate Angles A C B, and C B D, are equal to one another. Wherefore A B C, C B D, are two Triangles, having two Angles A B C, B C A, of the one, equal to two Angles B C D, C B D, of the other, each to each; and likewiſe one Side of the one equal to one Side of the other, *viz.* the Side B C between the equal Angles, which is common. Therefore the remaining Sides ſhall be † equal to the remaining Sides, each to each, and the remaining Angle to the remaining Angle. And ſo the Side A B is equal to the Side C D, the Side A C to B D, and the Angle B A C to the Angle B D C. And becauſe the Angle A B C is equal to the Angle B C D, and the Angle C B D to the Angle A C B; therefore the whole Angle A B D is equal to the whole Angle A C D: But it has been proved, that the Angle B A C is alſo equal to the Angle B D C.

* 29 of this.

† 26 of this.

Wherefore, *the oppoſite Sides and Angles of any Parallelogram are equal between themſelves.*

I ſay, moreover, that the Diameter biſects it. For becauſe A B is equal to C D, and B C is common, the two Sides A B, B C, are each equal to the two Sides D C, C B; and the Angle A B C is alſo equal to the Angle D C B. Therefore the Baſe A C is ‡ equal

‡ 4 of this.

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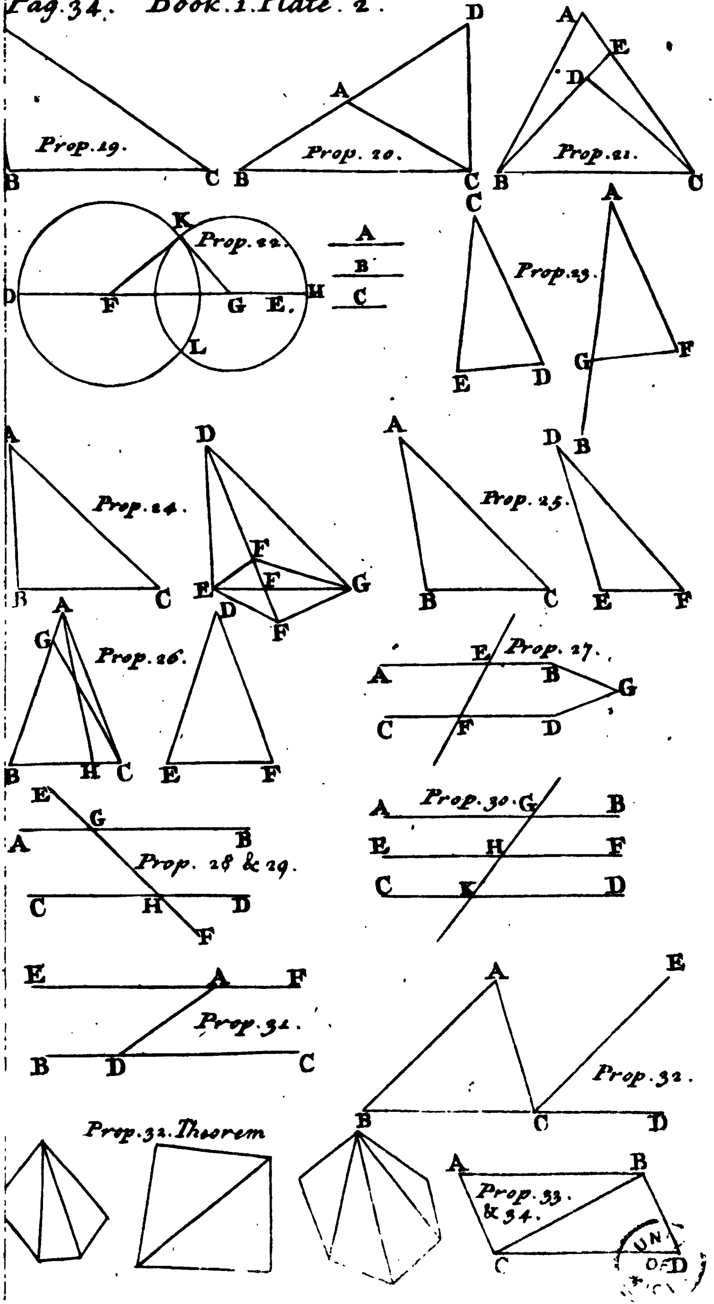


equal to the Baſe D B; and the Triangle A B C is equal to the Triangle B C D. Wherefore, *the Diameter* B C *biſects the Parallelogram* A C D B; which was to be demonſtrated.

PROPOSITION XXXV.

Theorem.

Parallelograms conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves.

LET A B C D, E B C F, be Parallelograms conſtituted upon the ſame Baſe B C, and between the ſame Parallels A F and B C. I ſay, the Parallelogram A B C D is equal to the Parallelogram E B C F.

For, becauſe A B C D is a Parallelogram, A D is
* equal to B C; and for the ſame Reaſon E F is equal * 34 *of this.*
to B C; wherefore A D ſhall be † equal to E F; but † *Ax.* 1.
D E is common. Therefore the whole A E is ‡ equal ‡ *Ax.* 2.
to the whole D F. But A B is equal to D C; wherefore E A, A B, the two Sides of the Triangle A B E, are equal to the two Sides F D, D C, each to each;
and the Angle F D C * equal to the Angle E A B, the * 29 *of this.*
outward one to the inward one. Therefore the Baſe
E B is † equal to the Baſe C F, and the Triangle E A B † 4 *of this.*
to the Triangle F D C. If the common Triangle
D G E be taken from both, there will remain ‡ the ‡ *Ax.* 3.
Trapezium A B G D, equal to the Trapezium F C G E; and if the Triangle G B C, which is common, be added, the Parallelogram A B C D will be equal to the Parallelogram E B C F. Therefore, *Parallelograms conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XXXVI.

THEOREM.

Parallelograms conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves.

LET the Parallelograms ABCD, EFGH, be conſtituted upon the equal Baſes BC, FG, and between the ſame Parallels AH, BG. I ſay, the Parallelogram ABCD is equal to the Parallelogram EFGH.

* *Hyp.* For join BE, CH. Then becauſe BC is * equal to FG, and FG to EH; BC will be likewiſe equal to EH; and they are parallel, and BE, CH, join them. But two Right Lines joining Right Lines, which are equal and parallel, towards the ſame Parts, † 33 *of this.* are † equal and parallel: Wherefore EBCH is a Pa- ‡ 35 *of this.* rallelogram, and is ‡ equal to the Parallelogram ABCD; for it has the ſame Baſe BC, and is conſtituted between the ſame Parallels BC, AH. For the ſame Reaſon, the Parallelogram EFGH is equal to the ſame Parallelogram EBCH. Therefore the Parallelogram ABCD ſhall be equal to the Parallelogram EFGH. And ſo *Parallelograms conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XXXVII.

THEOREM.

Triangles conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves.

LET the Triangles ABC, DBC, be conſtituted upon the ſame Baſe BC, and between the ſame Parallels AD, BC. I ſay, the Triangle ABC is equal to the Triangle DBC.

For produce AD both Ways to the Points E and * 31 *of this.* F; and through B draw * BE parallel to CA; and through C, CF, parallel to BD.

Wherefore both E B C A, D B C F, are Parallelograms; and the Parallelogram E B C A is * equal to * 35 of this. the Parallelogram D B C F; for they ſtand upon the ſame Baſe B C, and between the ſame Parallels B C, E F. But the Triangle A B C is † one half of the Parallelogram † 34 of this. E B C A, becauſe the Diameter A B biſects it; and the Triangle D B C is one half of the Parallelogram D B C F, for the Diameter D C biſects it. But Things that are the Halves of equal Things, are ‡ ‡ Ax. 7. equal between themſelves. Therefore the Triangle A B C is equal to the Triangle D B C. Wherefore, *Triangles conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XXXVIII.

Theorem.

Triangles conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves.

LET the Triangles A B C, D C E, be conſtituted upon the equal Baſes B C, C E, and between the ſame Parallels B E, A D. I ſay, the Triangle A B C is equal to the Triangle D C E.

For, produce A D both Ways to the Points, G, H; through B draw * B G parallel to C A; and * 31 of this. through E, E H, parallel to D C.

Wherefore both G B C A, D C E H, are Parallelograms; and the Parallelogram G B C A is † equal † 36 of this. to the Parallelogram D C E H: For they ſtand upon equal Baſes, B C, C E, and between the ſame Parallels B E, G H. But the Triangle A B C is ‡ one half ‡ 34 of this. of the Parallelogram G B C A, for the Diameter A B biſects it; and the Triangle D C E ‡ is one half of the Parallelogram D C E H, for the Diameter D E biſects it. But Things that are the Halves of equal Things, are * equal between themſelves. Therefore * Ax. 7. the Triangle A B C is equal to the Triangle D C E. Wherefore, *Triangles conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XXXIX.

THEOREM.

Equal Triangles conſtituted upon the ſame Baſe, on the ſame Side, are in the ſame Parallels.

LET A B C, D B C, be equal Triangles, conſtituted upon the ſame Baſe B C, on the ſame Side. I ſay they are between the ſame Parallels. For, let A D be drawn. I ſay, A D is parallel to B C.
* 31 *of this.* For, if it be not parallel, draw * the Right Line A E thro' the Point A, parallel to B C, and draw E C.
† 37 *of this.* Then the Triangle A B C † is equal to the Triangle E B C; for it is upon the ſame Baſe B C, and between the ſame Parallels B C, A E. But the Triangle A B C
‡ *From Hyp.* is ‡ equal to the Triangle D B C. Therefore the Triangle D B C is alſo equal to the Triangle E B C, a greater to a leſs, which is impoſſible. Wherefore A E is not parallel to B C: And by the ſame Way of Reaſoning we prove, that no other Line but A D is parallel to B C. Therefore A D is parallel to B C. Wherefore, *equal Triangles conſtituted upon the ſame Baſe, on the ſame Side, are in the ſame Parallels*; which was to be demonſtrated.

PROPOSITION XL.

THEOREM.

Equal Triangles conſtituted upon equal Baſes, on the ſame Side, are between the ſame Parallels.

LET A B C, C D E, be equal Triangles, conſtituted upon equal Baſes B C, C E. I ſay, they are between the ſame Parallels. For, let A D be drawn. I ſay, A D is parallel to B E.
* 31 *of this.* For, if it be not, let A F be drawn * through A, parallel to B E, and draw F E.
† 38 *of this.* Then the Triangle A B C is † equal to the Triangle F C E; for they are conſtituted upon equal Baſes, and between the ſame Parallels B E, A F. But the Triangle A B C is equal to the Triangle D C E. Therefore

ſore the Triangle D C E ſhall be equal to the Triangle F C E, the greater to the leſs, which is impoſſible. Wherefore A F is not parallel to B E. And in this manner we demonſtrate, that no Right Line can be parallel to B E, but A D. Therefore A D is parallel to B E. And ſo, *equal Triangles conſtituted upon equal Baſes, on the ſame Side, are between the ſame Parallels*; which was to be demonſtrated.

PROPOSITION XLI.

THEOREM.

If a Parallelogram and a Triangle have the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double to the Triangle.

LET the Parallelogram A B C D, and the Triangle E B C, have the ſame Baſe, and be between the ſame Parallels, B C, A E. I ſay, the Parallelogram A B C D is double the Triangle E B C.

For join A C.

Now the Triangle A B C is * equal to the Triangle E B C; for they are both conſtituted upon the ſame Baſe B C, and between the ſame Parallels B C, A E. But the Parallelogram A B C D is † double the Triangle A B C, ſince the Diameter A C biſects it. Wherefore likewiſe it ſhall be ‡ double to the Triangle E B C. *If*, therefore, *a Parallelogram and Triangle have both the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double the Triangle*; which was to be demonſtrated.

* 37 of this.
† 34 of this.
‡ Ax. 6.

PROPOSITION XLII.

THEOREM.

To conſtitute a Parallelogram equal to a given Triangle, in an Angle equal to a given Right-lined Angle.

LET the given Triangle be A B C, and the Right-lined Angle D. It is required to conſtitute a Parallelogram equal to the given Triangle A B C, in a Right-lined Angle equal to D.

* 10 *of this.* Biſect * B C in E, join A E, and at the Point E,
† 23 *of this.* in the Right Line E C, conſtitute † an Angle C E F
‡ 31 *of this.* equal to D. Alſo draw ‡ A G thro' A, parallel to
E C, and thro' C the Right Line C G, parallel to F E.

Now F E C G is a Parallelogram: And becauſe
* 38 *of this.* B E is equal to E C, the Triangle A B E ſhall be *
equal to the Triangle A E C; for they ſtand upon
equal Baſes B E, E C, and are between the ſame Parallels
B C, A G. Wherefore the Triangle A B C is
double to the Triangle A E C. But the Parallelo-
† 41 *of this.* gram F E C G is alſo † double to the Triangle A E C;
for it has the ſame Baſe, and is between the ſame Parallels.
Therefore the Parallelogram F E C G is
equal to the Triangle A B C, and has the Angle C E F
equal to the Angle D. Wherefore, *the Parallelogram*
F E C G *is conſtituted equal to the given Triangle* A B C,
in an Angle C E F *equal to a given Angle* D; which
was to be done.

PROPOSITION XLIII.

Theorem.

In every Parallelogram, the Complements of the Parallelograms, that ſtand about the Diameter, are equal between themſelves.

LET A B C D be a Parallelogram, whoſe Diameter is D B; and let F H, E G, be Parallelograms ſtanding about the Diameter B D. Now A K, K C, are called the Complements of them: I ſay, the Complement A K is equal to the Complement K C.

For ſince A B C D is a Parallelogram, and B D is
* 34 *of this.* the Diameter thereof, the Triangle A B D is * equal
to the Triangle B D C. Again, becauſe H K F D is
a Parallelogram, whoſe Diameter is D K, the Triangle
H D K ſhall * be equal to the Triangle D F K;
and for the ſame Reaſon the Triangle K B G is equal
to the Triangle K E B. But ſince the Triangle B E K
is equal to the Triangle B G K, and the Triangle
H D K to D F K, the Triangle B E K, together with
the Triangle H D K, is equal to the Triangle B G K,
together with the Triangle D F K. But the whole
Triangle A B D is likewiſe equal to the whole Triangle
B D C.

B D C. Wherefore the Complement remaining, A K, will be equal to the remaining Complement K C. Therefore, *in every Parallelogram, the Complements of the Parallelograms that ſtand about the Diameter, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XLIV.

Problem.

To apply a Parallelogram to a given Right Line, equal to a given Triangle, in a given Right-lined Angle.

LET the Right Line given be A B, the given Triangle C, and the given Right-lined Angle D. It is required to the given Right Line A B, to apply a Parallelogram equal to the given Triangle C, in an Angle equal to D.

Make the Parallelogram B E F G equal to the * Triangle C, in the Angle E B G, equal to D. Place B E in a ſtrait Line with A B, and produce F G to H, and through A let A H be drawn † parallel to either G B, or F E, and join H B. * 42 *of this.* † 32 *of this.*

Now, becauſe the Right Line H F falls on the Parallels A H, E F, the Angles A H F, H F E, are ‡ equal to two Right Angles. And ſo B H F, H F E, are leſs than two Right Angles; but Right Lines making leſs than two Right Angles, with a third Line, being infinitely produced, will meet * each other. Wherefore H B, F E, produced, will meet each other; which let be in K, through which * draw K L parallel to E A, or F H, and produce A H, G B, to the Point L and M. ‡ 29 *of this.* * *Ax.* 12. * 31 *of this.*

Therefore H L K F is a Parallelogram, whoſe Diameter is H K; and A G, M E, are Parallelograms about H K; whereof L B, B F are the Complements. Therefore L B is † equal to B F. But B F is alſo equal to the Triangle C. Wherefore likewiſe L B ſhall be equal to the Triangle C; and becauſe the Angle G B E is ‡ equal to the Angle A B M, and alſo equal to the Angle D, the Angle A B M ſhall be equal to the Angle D. Therefore, *to the given Right Line* A B *is applied a Parallelogram, equal to the given Triangle* C, *and the Angle* A B M, *equal to the given Angle* D; which was to be done. † 43 *of this.* ‡ 15 *of this.*

PROPOSITION XLV.

Problem.

To make a Parallelogram equal to a given Right-lined Figure, in a given Right-lined Angle.

LET A B C D be the given Right-lined Figure, and E the Right-lined Angle given. It is required to make a Parallelogram equal to the Right-lined Figure A B C D, in an Angle equal to E.

* 24 *of this.* Let D B be joined, and make * the Parallelogram
F H equal to the Triangle A D B, in an Angle H K F,
equal to the given Angle E,

† 44 *of this.* Then to the Right Line G H apply † the Parallelogram
G M, equal to the Triangle D B C in an Angle
G H M, equal to the Angle E.

Now, becauſe the Angle E is equal to H K F, or
G H M, the Angle H K F ſhall be equal to G H M,
add K H G to both; and the Angles H K F, K H G,
are, together, equal to the Angles K H G, G H M.
‡ 29 *of this.* But H K F, K H G, are ‡, together, equal to two
Right Angles. Wherefore, likewiſe, the Angles
K H G, G H M, ſhall be equal to two Right Angles;
And ſo, at the given Point H in the Right Line G H,
two Right Lines K H, H M, not drawn on the ſame
Side, make the adjacent Angles, both together, equal
* 14 *of this.* to two Right Angles; and conſequently K H, H M *,
make one ſtrait Line. And becauſe the Right Line
H G falls upon the Parallels K M, F G, the alternate
Angles M H G, H G F, are ‡ equal. And if H G L
be added to both, the Angles M H G, H G L, together,
are equal to the Angles H G F, H G L, together,
* 29 *of this.* But the Angles M H G, H G L, are * together equal
to two Right Angles. Wherefore, likewiſe, the Angles
H G F, H G L, are together equal to two Right
Angles; and ſo, F G, G L, make one ſtrait Line.
And ſince K F is equal and parallel to H G, as likewiſe
† 30 *of this.* H G to M L, K F ſhall be † equal and parallel
to M L, and the Right Lines K M, F L, join them.
‡ 33 *of this.* Wherefore K M, F L, are ‡ equal and parallel. Therefore
K F L M is a Parallelogram. But ſince the Triangle
A B D is equal to the Parallelogram H F, and
the

the Triangle D B C to the Parallelogram G M; therefore the whole Right-lined Figure A B C D will be equal to the whole Parallelogram K F L M. Therefore, *the Parallelogram* K F L M *is made equal to the given Right-lined Figure* A B C D, *in an Angle* F K M, *equal to the given Angle* E; which was to be done.

Coroll. It is manifeſt from what has been ſaid, how to apply a Parallelogram to a given Right Line, equal to a given Right-lined Figure in a given Right-lined Angle.

PROPOSITION XLVI.

PROBLEM.

To deſcribe a Square upon a given Right Line.

LET A B be the Right Line given, upon which it is required to deſcribe a Square.

Draw * A C at Right Angles to A B from the Point * 11 *of this.*
A given therein; make † A D equal to A B, and thro' † 3 *of this.*
the Point D draw ‡ D E parallel to A B; alſo thro' ‡ 31 *of this.*
B draw B E parallel to A D.

Then A D E B is a Parallelogram; and ſo A B * is * 34 *of this.*
equal to D E, and A D to B E. But B A is equal to A D. Therefore the four Sides B A, A D, B E, E D, are equal to each other.

And ſo the Parallelogram A D E B is equilateral: I ſay, it is likewiſe equiangular. For, becauſe the Right Line A D falls upon the Parallels A B, D E, the Angles B A D, A D E, are † equal to two Right Angles. † 29 *of this.*
But, B A D is a Right Angle. Wherefore A D E is alſo a Right Angle; but the oppoſite Sides and oppoſite Angles of Parallelograms are ‡ equal. Therefore, ‡ 34 *of this.*
each of the oppoſite Angles A B E, B E D, are Right Angles; and conſequently A D B E is a Rectangle: But it has been proved to be equilateral. Therefore, *it is neceſſarily a * Square, and is deſcribed upon the* * Def. 30.
Right Line A B; which was to be done.

Coroll. Hence every Parallelogram that has one Right Angle, is a Rectangle.

PRO-

PROPOSITION XLVII.

THEOREM.

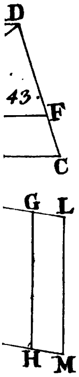
In any Right-angled Triangle, the Square deſcribed upon the Side, ſubtending the Right Angle, is equal to both the Squares, deſcribed upon the Sides, containing the Right Angle.

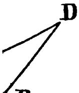
LET A B C be a Right-angled Triangle, having the Right Angle B A C. I ſay, the Square, deſcribed upon the Right Line B C, is equal to both the Squares, deſcribed upon the Sides B A, A C.

* 46 *of this.* For, deſcribe * upon B C the Square B D E C, and
on B A, A C, the Squares G B, H C; and through
‖ 31 *of this.* the Point A draw A L parallel to ‖ B D, or C E; and
let A D, F C, be joined.

† *Def.* 30. Then, becauſe the Angles B A C, B A G, † are
Right ones, two Right Lines A G, A C, at the given
Point A, in the Right Line B A, being on contrary
Sides thereof, make the adjacent Angles equal to two
‡ 14 *of this.* Right Angles. Therefore C A, A G, make ‡ one
ſtrait Line, by the ſame Reaſon A B, A H, make one
ſtrait Line. And ſince the Angle D B C is equal to
the Angle F B A, for each of them is a Right one,
add A B C, which is common, and the whole Angle
* *Ax.* 2. D B A is * equal to the whole Angle F B C. And
ſince the two Sides A B, B D, are equal to the two
Sides F B, B C, each to each, and the Angle D B A
† 4 *of this.* equal to the Angle F B C; the Baſe A D will be †
equal to the Baſe F C, and the Triangle A B D equal
to the Triangle F B C: But the Parallelogram B L
is ‡ double to the Triangle A B D; for they have the
ſame Baſe D B, and are between the ſame Parellels
‡ 41 *of this.* BD, A L. The Square G B is ‡ alſo double to the
Triangle F B C; for they have the ſame Baſe F B,
and are in the ſame Parallels F B, G C. But Things
* *Ax.* 6. that are the Doubles of equal Things, are * equal to
each other. Therefore the Parallelogram B L is equal
to the Square G B. After the ſame manner, A E,
B K, being joined, we prove that the Parallelogram
C L is equal to the Square H C. Therefore the whole
Square B D E C is equal to the two Squares G B,
H C. But the Square B D E C is deſcribed on the
Right

Prop. 41.





Prop. 48.

UNIV. OF MICH.

Right Line B C, and the Squares G B, H C, on B A, A C. Therefore the Square B E, deſcribed on the Side B C, is equal to the Squares deſcribed on the Sides B A, A C. Wherefore, *in any Right-angled Triangle, the Square deſcribed upon the Side, ſubtending the Right Angle, is equal to both the Squares deſcribed upon the Sides, containing the Right Angle*; which was to be demonſtrated.

PROPOSITION XLVIII.

THEOREM.

If a Square deſcribed upon one Side of a Triangle, be equal to the Squares deſcribed upon the other two Sides of the ſaid Triangle; then, the Angle contained by thoſe two other Sides is a Right Angle.

IF the Square, deſcribed upon the Side B C of the Triangle A B C, be equal to the Squares, deſcribed upon the other two Sides of the Triangle B A, A C; I ſay, the Angle B A C is a Right one.

For, let there be drawn A D from the Point A, at Right Angles, to A C: Likewiſe make A D equal to B A, and join D C.

Then, becauſe D A is equal to A B, the Square deſcribed on D A will be equal to the Square deſcribed on A B. And adding the common Square deſcribed on A C, the Squares deſcribed on D A, A C, are equal to the Squares deſcribed on B A, A C. But the Square deſcribed on D C is * equal to the Squares deſcribed on D A, A C; for D A C is a Right Angle: But the Square on B C is put equal to the Squares on B A, A C. Therefore the Square deſcribed on D C is equal to the Square deſcribed on B C; and ſo the Side C D is equal to the Side C B. And becauſe D A is equal to A B, and A C is common, the two Sides D A, A C, are equal to the two Sides B A, A C; and the Baſe D C is equal to the Baſe C B. Therefore the Angle D A C is ‡ equal to the Angle B A C; but D A C is a Right Angle; and ſo B A C will be a Right Angle alſo. *If*, therefore, *a Square deſcribed upon one Side of a Triangle, be equal to the Squares, deſcribed upon the other two Sides of the ſaid Triangle, then the Angle, contained by theſe two other Sides, is a Right Angle*; which was to be demonſtrated.

* 47 *of this.*

‡ 8 *of this.*

EUCLID's

EUCLID's

ELEMENTS.

BOOK II.

DEFINITIONS.

I. *EVERY Right-angled Parallelogram is ſaid to be contained under two Right Lines, comprehending a Right Angle.*

II. *In every Parallelogram, either of thoſe Parallelograms, that are about the Diameter, together with the Complements, is called a Gnomon.*

PRO-

PROPOSITION I.

THEOREM.

If there be two Right Lines, and one of them be divided into any Number of Parts; the Rectangle comprehended under the whole Line and the divided Line, ſhall be equal to all the Rectangles contained under the whole Line, and the ſeveral Segments of the divided Line.

LET A and B C be two Right Lines, whereof B C is cut or divided any how in the Points D, E. I ſay, the Rectangle contained under the Right Lines A and B C, is equal to the Rectangles contained under A and B D, A and D E, and A and E C.

For, let * B F be drawn from the Point B, at Right * 11. 1.
Angles, to B C; and make † B G equal to A; and † 3. 1.
let ‡ G H be drawn thro' G parallel to B C: Like- ‡ 31. 1.
wiſe let ‡ there be drawn D K, E L, C H, through D, E, C, parallel to B G.

Then the Rectangle B H, is equal to the Rectangles B K, D L, and E H; but the Rectangle B H is that contained under A and B C; for it is contained under G B, B C; and G B is equal to A; and the Rectangle B K is that contained under A and B D; for it is contained under G B and B D; and G B is equal to A; and the Rectangle D L is that contained under A and D E, becauſe D K, that is, B G, is equal to A: So likewiſe the Rectangle E H is that contained under A and E C. Therefore the Rectangle under A and B C is equal to the Rectangles under A and B D, A and D E, and A and E C. Therefore, *if there be two Right Lines given, and one of them be divided into any Number of Parts, the Rectangle comprehended under the whole Line and the divided Line, ſhall be equal to all the Rectangles contained under the whole Line, and the ſeveral Segments of the divided Line*; which was to be demonſtrated.

PROPOSITION II.

Theorem.

If a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.

LET the Right Line A B be any how divided in the Point C. I ſay, the Rectangle contained under A B and B C, together with that contained under A B, and A C, is equal to the Square made on A B.

*46. 1.

For let the Square A D E B be deſcribed * on A B; and thro' C let C F be drawn parallel to A D or B E. Therefore A E is equal to the Rectangles A F and C E. But A E is a Square deſcribed upon A B; and A F is the Rectangle contained under B A and A C; for it is contained under D A and A C, whereof A D is equal to A B; and the Rectangle C E is contained under A B and B C, ſince B E is equal to A B. Wherefore the Rectangle under A B and A C, together with the Rectangle under A B and B C, is equal to the Square of A B. Therefore, *if a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line*; which was to be demonſtrated.

PROPOSITION III.

Theorem.

If a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts, together with the Square of the firſt-mentioned Part.

LET the Right Line A B be any how cut in the Point C. I ſay, the Rectangle under A B and B C is equal to the Rectangle under A C and B C, together with the Square deſcribed on B C.

For deſcribe * the Square C D E B upon B C; * 46. 1.
produce E D to F; and let A F be drawn † thro' A, † 31. 1.
parallel to C D or B E.

Then the Rectangle A E ſhall be equal to the two Rectangles A D, C E: And the Rectangle A E is that contained under A B and B C; for it is contained under A B and B E, whereof B E is equal to B C: And the Rectangle A D is that contained under A C and C B, ſince D C is equal to C B: And D B is a Square deſcribed upon B C. Wherefore the Rectangle under A B and B C is equal to the Rectangle under A C and C B, together with the Square deſcribed upon B C. Therefore, *if a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts, together with the Square of the firſt-mentioned Part*; which was to be demonſtrated.

PROPOSITION IV.

THEOREM.

If a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments.

LET the Right Line A B be any how cut in C. I ſay, the Square made on A B is equal to the Squares of A C, C B, together with twice the Rectangle contained under A C, C B.

For * deſcribe the Square A D E B upon A B, join * 46. 1.
B D, and thro' C draw † C G F parallel to A D or B E; † 31. 1.
and alſo thro' G draw H K parallel to A B or D E.

Then, becauſe C F is parallel to A D, and B D falls
upon them, the outward Angle B G C ſhall be ‡ equal ‡ 29. 1.
to the inward and oppoſite Angle A D B; but the
Angle A D B is * equal to the Angle A B D, ſince the * 5. 1.
Side B A is equal to the Side A D. Wherefore the
Angle C G B is equal to the Angle G B C; and ſo
the Side B C equal † to the Side C G; but likewiſe † 6. 1.
the Side C B is ‡ equal to the Side G K, and the Side ‡ 34. 1.
C G to B K. Therefore G K is equal to K B, and

C G K B is equilateral. I ſay, it is alſo Right-angled; for, becauſe C G is parallel to B K, and C B falls on † 29. 1. them, the Angles K B C, G C B †, are equal to two Right Angles. But K B C is a Right Angle. Wherefore G C B alſo is a Right Angle, and the oppoſite Angles C G K, G K B, ſhall be Right Angles, Therefore C G K B is a Rectangle. But it has been proved to be equilateral. Therefore C G K B is a Square deſcribed upon B C. For the ſame Reaſon H F is alſo a Square made upon H G, and (becauſe H G is † 34. 1. equal to A C †) it is equal to the Square of A C. Wherefore H F and C K are the Squares of A C and * 43. 1. C B. And, becauſe the Rectangle A G is * equal to the Rectangle G E, and A G is that which is contained under A C and C B; for G C is equal to C B; therefore G E ſhall be equal to the Rectangle under A C, and C B. Wherefore the Rectangles A G, and G E, are equal to twice the Rectangle contained under A C, and C B; and H F and C K, are the Squares of A C, C B. Therefore the four Figures H F, C K, A G, G E, are equal to the Squares of A C and C B, with twice the Rectangle contained under A C and C B. But H F, C K, A G, G E, make up the whole Square of A B, *viz.* A D E B. Therefore the Square of A B is equal to the Squares of A C and C B, together with twice the Rectangle contained under A C and C B. Wherefore, *if a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segment;* which was to be demonſtrated.

Coroll. Hence it is manifeſt, that the Parallelograms which ſtand about the Diameter of a Square, are likewiſe Squares.

PROPOSITION V.

THEOREM.

If a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Diſtance, is equal to the Square made of half the Line.

LET any Right Line A B be cut into two equal Parts in C, and into two unequal Parts in D. I ſay, the Rectangle contained under A D, and D B, together with the Square of C D, is equal to the Square of B C.

For † deſcribe C E F B, the Square of B C; draw † B E, and through D draw * D H G, parallel to C E, or B F; and through H draw K L O, parallel to C B, or E F; and A K through A, parallel to C L, or B O. † 46. 1. * 31. 1.

Now the Complement C H is ‡ equal to the Complement H F. Add D O, which is common to both of them, and the whole C O is equal to the whole D F: But C O is equal to A L, becauſe A C is equal to C B †; therefore A L is equal to D F; and, adding C H, which is common, the whole A H ſhall be equal to F D, D L, together. But A H is the Rectangle contained under A D, and D B; for D H is * equal to D B, and F D, D L, is the Gnomon M N X; therefore M N X is equal to the Rectangle contained under A D, and D B; and if L G, being common, and equal to the Square of C D, be added, then the Gnomon M N X, and L G, are equal to the Rectangle contained under A D, and D B, together with the Square of C D; but the Gnomon, M N X, and L G, make up the whole Square C E F B, *viz.* the Square of C B. Therefore the Rectangle under A D, and D B, together with the Square of C D, is equal to the Square of C B. Wherefore, *if a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Diſtance, is equal to the Square made of half the Line*; which was to be demonſtrated. ‡ 43. 1. † 36. 1. * *Cor.* 4. *of this.*

PROPOSITION VI.

THEOREM.

If a Right Line be divided into two equal Parts, and another Right Line be added directly to the ſame, the Rectangle contained under [the Line compounded of] the whole and added Line (taken as one Line) and the added Line, together with the Square of half the firſt Line, is equal to the Square of [the Line compounded of] half the Line and the added Line, taken as one Line.

LET the Right Line A B be biſected in the Point C, and B D added directly thereto. I ſay, the Rectangle under A D, and B D, together with the Square of B C, is equal to the Square of C D.

* 46. 1. For, deſcribe * C E F D, the Square of C D, and
† 31. 1. join D E; draw † B H G thro' B, parallel to C E, or
D F, and K L M thro' H, parallel to A D, or E F, as
alſo A K thro' A, parallel to C L, or D M.

Then becauſe A C is equal to C B, the Rectangle
‡ 36. 1. A L ſhall be ‡ equal to the Rectangle C H; but C H
* 43. 1. is * equal to H F. Therefore A L will be equal to
H F; and adding C M, which is common to both,
then the whole Rectangle A M is equal to the Gnomon
N X O. But A M is that Rectangle which is
† Cor. 4. of contained under A D, and D B; for D M is † equal
this. to D B; therefore the Gnomon N X O is equal to
the Rectangle under A D, and D B. Add L G,
‡ Cor. 4. of which is common, *viz.* ‡ the Square of C B; and then
this. the Rectangle under A D, D B, together with the
Square of B C, is equal to the Gnomon N X O
with L G. But the Gnomon N X O, and L G, together, make up the Figure C E F D, that is, the Square of C D. Therefore the Rectangle under A D, and D B, together with the Square of B C, is equal to the Square of C D. Therefore, *if a Right Line be divided into two equal Parts, and another Right Line be added directly to the ſame, the Rectangle contained under [the Line compounded of] the whole and added Line (taken as one Line) and the*

added Line, together with the Square of half the firſt Line, is equal to the Square of [the Line compounded of] half the Line and the added Line, taken as one Line; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the ſaid Segment, together with the Square made of the other Segment.

LET the Right Line A B be any how cut in the Point C. I ſay, the Squares of A B, and B C, together, are equal to double the Rectangle contained under A B, and B C, together with the Square, made of A C.

For let the Square of A B be * deſcribed, *viz.* A D E B, and conſtruct § the Figure. * 46. 1.

Then, becauſe the Rectangle A G is † equal to the Rectangle G E; if C F, which is common, be added to both, the whole Rectangle A F ſhall be equal to the whole Rectangle C E; and ſo the Rectangles A F, C E, taken together, are double to the Rectangle A F; but A F, C E, make up the Gnomon K L M, and the Square C F. Therefore the Gnomon K L M, together with the Square C F, ſhall be double to the Rectangle A F. But double the Rectangle under A B, and B C, is double the Rectangle A F; for B F is ‡ equal to B C. Therefore the Gnomon K L M, and the Square C F, are equal to double the Rectangle contained under A B, and B C. And if H F, which is common, being the Square of A C, be added to both, then the Gnomon K L M, and the Squares C F, H F,

† 43. 1.
‡ Cor. 4.

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§ *A Figure is ſaid to be conſtructed, when Lines drawn in a Parallelogram, parallel to the Sides thereof, cut the Diameter in one Point, and make two Parallelograms about the Diameter, and two Complements. So likewiſe a double Figure is ſaid to be conſtructed, when two Right Lines, parallel to the Sides, make four Parallelograms about the Diameter, and four Complements.*

are equal to double the Rectangle contained under A B, and B C, together with the Square of A C. But the Gnomon K L M, together with the Squares C F, H F, are equal to A D E B, and C F, *viz.* the Squares of A B, B C. Therefore the Squares of A B, and B C, are together equal to double the Rectangle contained under A B, and B C, together with the Square of A C. Therefore, *if a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the ſaid Segment, together with the Square, made of the other Segment*; which was to be demonſtrated.

PROPOSITION VIII.

THEOREM.

If a Right Line be any how cut into two Parts, four Times the Rectangle, contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of [the Line compounded of] the whole Line and the firſt Part, taken as one Line.

LET the Right Line A B be cut any how in C. I ſay, four Times the Rectangle contained under A B, and B C, together with the Square of A C, is equal to the Square of A B, and B C, taken as one Line.

For, let the Right Line A B be produced to D, ſo that B D be equal to B C; deſcribe the Square A E F D, on A D, and conſtruct the double Figure.

* Hyp. † 34. 1. Now, ſince C B is * equal to B D, and alſo to † G K, and B D is equal to K N; therefore G K ſhall be likewiſe equal to K N: By the ſame Reaſoning, P R is equal to R O. And ſince C B is equal to B D, ‡ 36. 1. and G K to K N, the Rectangle C K will ‡ be equal to the Rectangle B N, and the Rectangle G R to the ‖ 43. 1. Rectangle R N. But C K is ‖ equal to R N; for they are the Complements of the Parallelogram C O. Therefore B N is equal to G R, and the four Squares B N, C K, G R, R N, are equal to each other; and ſo they are together quadruple C K. Again, becauſe C B is equal to B D, and B D to B K, that is, equal

to C G; and the ſaid C B is equal alſo to G K, that is, to G P; therefore C G ſhall be equal to G P. But P R is equal to R O; therefore the Rectangle A G ſhall be equal to the Rectangle M P, and the Rectangle P L equal to R F. But M P is equal to P L; for they are the Complements of the Parallelogram M L. Wherefore A G is equal alſo to R F. Therefore the four Parallelograms A G, M P, P L, R F, are equal to each other, and accordingly they are together quadruple of A G. But it has been proved, that the four Squares C K, B N, G R, R N, are quadruple of C K. Therefore the four Rectangles, and the four Squares, making up the Gnomon S T Y, are together quadruple of A K; and becauſe A K is a Rectangle contained under A B, and B C, for B K is equal to B C; therefore four Times the Rectangle under A B and B C, will be quadruple of A K. But the Gnomon S T Y has been proved to be quadruple of A K. And ſo four Times the Rectangle contained under A B, and B C, is equal to the Gnomon S T Y. And if X H, being equal to † the Square of A C, which is common, be added to both; then four Times the Rectangle contained under A B, and B C, together with the Square of A C, is equal to the Gnomon S T Y, and the Square X H. But the Gnomon S T Y and X H make A E F D, the whole Square of A D. Therefore four Times the Rectangle contained under A B, and B C, together with the Square of A C, is equal to the Square of A D, that is, of A B and B C taken as one Line. Wherefore, *if a Right Line be any how cut into two Parts, four Times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of [the Line compounded of] the whole Line and the firſt Part, taken as one Line*; which was to be demonſtrated.

† Cor. 4. of this.

PROPOSITION IX.

Theorem.

If a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts, together, are double to the Square of the half Line, and the Square of the intermediate Part.

LET any Right Line A B be cut unequally in D,
and equally in C. I ſay, the Squares of A D,
D B, together, are double to the Squares of A C and
C D together.

* 11. 1. For, let * C E be drawn from the Point C at Right
Angles to A B, which make equal to A C, or C B;
† 31. 1. and join E A, E B. Alſo through D let † D F be
drawn parallel to C E, and F G through F parallel to
A B, and draw A F.

Now, becauſe A C is equal to C E, the Angle
‡ 5. 1. E A C will be ‡ equal to the Angle A E C; and ſince
the Angle at C is a Right one, the other Angles,
* Cor. 3. 32. 1. A E C, E A C, together, ſhall * make one Right Angle,
and are equal to each other: And ſo A E C,
E A C, are each equal to half a Right Angle. For the
ſame Reaſons are alſo C E B, E B C, each of them half
a Right Angle. Therefore the whole Angle A E B is
a Right Angle. And ſince the Angle G E F is half a
† 29. 1. Right one, and E G F is a Right Angle; for it is †
equal to the inward and oppoſite Angle E C B; the
other Angle E F G will be alſo equal to half a Right
one. Therefore the Angle G E F is equal to the Angle
‡ 6. 1. E F G. And ſo the Side E G is ‡ equal to the
Side G F. Again, becauſe the Angle at B is half a
Right one, and F D B is a Right one, becauſe equal
to the inward and oppoſite Angle E C B, the other
Angle B F D will be half a Right Angle. Therefore
the Angle at B is equal to the Angle B F D; and ſo
the Side D F is equal to the Side D B. And becauſe
A C is equal to C E, the Square of A C will be equal
to the Square of C E. Therefore the Squares of A C,
and C E, together, are double to the Square of A C;
† 47. 1. but the Square of E A is † equal to the Squares of

A C, and C E, together, ſince A C E is a Right Angle. Therefore the Square of E A is double to the Square of A C. Again, becauſe E G is equal to G F, and the Square of E G is equal to the Square of G F; therefore the Squares of E G, G F, together, are double to the Square of G F. But the Square of E F is † equal to the Squares of E G, G F. Therefore † 47. 1 the Square of E F is double the Square of G F: But G F is equal to C D; and ſo the Square of E F is double to the Square of C D. But the Square of A E is likewiſe double to the Square of A C. Wherefore the Squares of A E, and E F, are double to the Squares of A C, and C D. But the Square of A F is † equal to the Squares of A E, and E F; becauſe the Angle A E F is a Right Angle, and conſequently the Square of A F is double to the Squares of A C, and C D. But the Squares of A D, and D F, are equal to the Square of A F: For the Angle at D is a Right Angle. Therefore the Squares of A D, and D F, together, ſhall be double to the Squares of A C, and C D, together. But D F is equal to D B. Therefore the Squares of A D, and D B, together, will be double to the Squares of A C, and C D, together. Wherefore, *if a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Squares of the half Line, and the Squares of the intermediate Part*; which was to be demonſtrated.

PROPOSITION X.

THEOREM.

If a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, ſhall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.

LET the Right Line A B be biſected in C, and any ſtrait Line B D added directly thereto. I ſay, the Squares of A D, and D B, together, are double to the Squares of A C, and C D, together.

For,

‡ 47. 1. E B. But the Squares of B A and A E are ‡ equal to the Square of E B; for the Angle at A is a Right Angle. Therefore the Rectangle under C F and F A, together with the Square of A E, is equal to the Squares of B A, and A E. And, taking away the Square of A E, which is common, the remaining Rectangle under C F and F A is equal to the Square of A B. But F K is the Rectangle under C F and F A; ſince A F is equal to F G; and the Square of A B is A D. Wherefore the Rectangle F K is equal to the Square A D. And if A K, which is common, be taken from both, then the remaining Square F H is equal to the remaining Rectangle H D. But H D is the Rectangle under A B and B H, ſince A B is equal to B D; and F H is the Square of A H. Therefore, *the Rectangle under* A B *and* B H *ſhall be equal to the Square of* A H. *And ſo the given Right Line* A B *is cut in* H, *ſo that the Rectangle under* A B *and* B H *is equal to the Square of* A H; which was to be done.

PROPOSITION XII.

THEOREM.

In an obtuſe-angled Triangle, the Square of the Side ſubtending the obtuſe Angle is greater than the Squares of the Sides containing the obtuſe Angle, by twice the Rectangle under one of the Sides, containing the obtuſe Angle, viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular and the obtuſe Angle.*

LET A B C be an obtuſe-angled Triangle, having the obtuſe Angle B A C; and * from the Point B
* 12. 1. draw B D perpendicular to the Side C A produced. I ſay, the Square of B C is greater than the Squares of B A, and A C, by twice the Rectangle contained under C A, and A D.

For, becauſe the Right Line C D is any how cut in
† 4 of this. the Point A, the Square of C D ſhall be † equal to the Squares of C A, and A D, together with twice the Rectangle under C A, and A D. And if the Square of B D, which is common, be added, then the Squares of

of C D and D B are equal to the Squares of C A, A D, and D B, and twice the Rectangle contained under C A and A D. But the Square of C B is * equal to the Squares of C D, D B; for the Angle at D is a Right one, ſince B D is perpendicular; and the Square of A B is * equal to the Squares of A D and D B. Therefore the Square of C B is equal to the Squares of C A, and A B, together with twice a Rectangle under C A, and A D. Therefore, *in an obtuſe-angled Triangle, the Square of the Side ſubtending the obtuſe Angle is greater than the Squares of the Sides containing the obtuſe Angle, by twice the Rectangle under one of the Sides containing the obtuſe Angle,* viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular and the obtuſe Angle;* which was to be demonſtrated.

* 47. 1.

PROPOSITION XIII.

Theorem.

In an acute-angled Triangle, the Square of the Side ſubtending the acute Angle is leſs than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. *that on which the Perpendicular falls, and the Line aſſumed within the Triangle, from the Perpendicular to the acute Angle.*

LET A B C be an acute-angled Triangle, having the acute Angle B; and from A let there * be drawn A D perpendicular to B C. I ſay, the Square of A C is leſs than the Squares of C B, and B A, by twice a Rectangle under C B, and B D,

* 12. 1.

For, becauſe the Right Line C B is cut any how in D, the Squares of C B, and B D will be † equal to twice a Rectangle under C B, and B D, together with the Square of D C. And if the Square of A D be added to both, then the Squares of C B, B D, and D A, are equal to twice the Rectangle contained under C B, and B D, together with the Squares of A D, and D C. But the Square of A B is ‡ equal to the Squares of B D, and D A; for the Angle at D is a Right

† 7 of this.

‡ 47. 1.

‡ 47. 1. Right Angle. And the Square of A C is ‡ equal to the Squares of A D and D C. Therefore the Squares of C B and B A are equal to the Square of A C, together with twice the Rectangle contained under C B and B D. Wherefore the Square of A C, only, is leſs than the Squares of C B and B A, by twice the Rectangle under C B and B D. Therefore, *in an acute-angled Triangle, the Square of the Side ſubtending the acute Angle is leſs than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. that on which the Perpendicular falls, and the Line aſſumed within the Triangle, from the Perpendicular to the acute Angle*; which was to be demonſtrated.

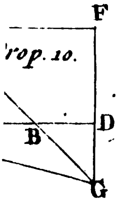
PROPOSITION XIV.

PROBLEM.

To make a Square equal to a given Right-lined Figure.

LET A be the given Right-lined Figure. It is required to make a Square equal thereto.

* 45. 1. Make * the Right-angled Parallelogram B C D E equal to the Right-lined Figure A. Now if B E be equal to E D, what was propoſed will be already done, ſince the Square B D is made equal to the Right-lined Figure A: But if it be not, let either B E or E D be the greater: Suppoſe B E, which let be produced to F; ſo that E F be equal to E D. This being done, let B F be † biſected in G, about which, as † 10. 1. a Centre, with the Diſtance G B, or G F, deſcribe the Semicircle B H F; and let D E be produced to H, and draw G H. Now, becauſe the Right Line B F is divided into two equal Parts in G, and into two unequal ones in E, the Rectangle under B E and E F, ‡ 5 of this. together with the Square of G E, ſhall be ‡ equal to the Square of G F. But G F is equal to G H. Therefore the Rectangle under B E and E F, together, with the Square of G E, is equal to the Square of G H. But * 47. 1. the Squares of H E and G E are * equal to the Square of G H. Wherefore the Rectangle under B E and E F,



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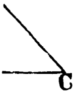

E F, together with the Square of G E, is equal to the Squares of H E and G E. And if the Square of E G, which is common, be taken from both, the remaining Rectangle, contained under B E and E F, is equal to the Square of E H. But the Rectangle under B E and E F is the Parallelogram B D, becauſe E F is equal to E D. Therefore the Parallelogram B D is equal to the Square of E H; but the Parallelogram B D is equal to the Right-lined Figure A. Wherefore the Right-lined Figure A is equal to the Square of E H. And ſo, *there is a Square made equal to the given Right-lined Figure* A, *viz. the Square of* E H; which was to be done.

The END *of the* SECOND BOOK.

EUCLID's

ELEMENTS.

BOOK III.

DEFINITIONS.

I. *EQUAL Circles are ſuch whoſe Diameters are equal; or from whoſe Centres the Right Lines that are drawn are equal.*

II. *A Right Line is ſaid to touch a Circle, when meeting the ſame, and being produced, it does not cut it.*

III. *Circles are ſaid to touch each other, which meeting do not cut one another.*

IV. *Right Lines in a Circle are ſaid to be equally diſtant from the Centre, when Perpendiculars drawn from the Centre to them are equal.*

V. *And that Line is ſaid to be farther from the Centre, on which the greater Perpendicular falls.*

VI. *A Segment of a Circle is a Figure contained under a Right Line, and a Part of the Circumference of a Circle.*

VII. *An Angle of a Segment is that which is contained by a Right Line, and the Circumference of a Circle.*

VIII. *An Angle is ſaid to be in a Segment, when ſome Point is taken in the Circumference thereof, and from it Right Lines are drawn to the Ends of that Right Line, which is the Baſe of the Segment; then the Angle contained under the Lines, ſo drawn, is ſaid to be an Angle in a Segment.*

IX. *But when the Right Lines containing the Angle do receive any Circumference of the Circle, then the Angle is ſaid to ſtand upon that Circumference.*

X. *A Sector of a Circle is that Figure, which is comprehended between two Right Lines, drawn from the Centre, and the Circumference contained between them.*

XI. *Similar Segments of Circles are thoſe, which include equal Angles, or whereof the Angles in them are equal.*

PROPOSITION I.

Problem.

To find the Centre of a Circle given.

LET A B C be the Circle given. It is required to find the Centre thereof.

Let the Right Line A B be any how drawn
in it, which * biſect in the Point D; and let D C be † * 10. 1.
drawn from the Point D, at Right Angles to A B, † 11. 1.
which let be produced to E.

Then, if E C be * biſected in F, I ſay, the Point F is the Centre of the Circle A B C.

For, if it be not, let G be the Centre, and let G A,
G D, G B, be drawn. Now, becauſe D A is equal to
D B, and D G is common, the two Sides A D, D G,
are equal to the two Sides G D, D B, each to each;
alſo the Baſe G A is ‡ equal to the Baſe G B, for they ‡ *Def.* 15. 1.
are drawn from the Centre G. Therefore the Angle
A D G is * equal to the Angle G D B. But when a * 8. 1.
Right Line ſtanding upon a Right Line makes the
adjacent Angles equal to one another, each of the
equal Angles will † be a Right Angle. Wherefore † *Def.* 10. 1.

the Angle G D B is a Right Angle. But F D B is alſo a Right Angle. Therefore, the Angle F D B is equal to the Angle G D B, a greater to a leſs, which is abſurd. Wherefore G is not the Centre of the Circle A B C. After the ſame manner we prove, that no other Point, unleſs F, is the Centre. Therefore, F *is the Centre of the Circle* A B C; which was to be found.

Coroll. If in a Circle any Right Line cuts another Right Line into two equal Parts and at Right Angles, the Centre of the Circle will be in that cutting Line.

PROPOSITION II.

Theorem.

If any two Points be aſſumed in the Circumference of a Circle, the Right Line joining thoſe two Points ſhall fall within the Circle.

LET A B C be a Circle; in the Circumference of which let any two Points A, B, be aſſumed. I ſay, a Right Line drawn, from the Point A, to the Point B, falls within the Circle.

* 1 of this. Find D * the Centre of the given Circle, and let any Point E be taken in the Right Line A B, and let D A, D E, D B, be joined.

Then becauſe D A is equal to D B, the Angle
† 5. 1. D A B will be † equal to the Angle D B A; and ſince the Side A E of the Triangle D A E is produced, the
‡ 16. 1. Angle D E B will be ‡ greater than the Angle D A E. But the Angle D A E is equal to the Angle D B E; therefore the Angle D E B is greater than the Angle D B E. But the greater Angle ſubtends the greater
|| 19. 1. Side. Wherefore D B || is greater than D E. But D B only comes to the Circumference of the Circle; therefore D E does not reach ſo far. And ſo the Point E falls within the Circle. Therefore, *if two Points are aſſumed in the Circumference of a Circle, the Right Line joining thoſe two Points ſhall fall within the Circle*; which was to be demonſtrated.

Coroll. Hence if a Right Line touches a Circle, it will touch it in one Point only.

PRO-

PROPOSITION III.

Theorem.

If in a Circle, a Right Line drawn thro' the Centre cuts any other Right Line, not drawn thro' the Centre, into equal Parts, it ſhall cut it at Right Angles; and if it cuts it at Right Angles, it ſhall cut it into two equal Parts.

LET ABC be a Circle, wherein the Right Line CD, drawn thro' the Centre, biſects the Right Line AB, not drawn thro' the Centre. I ſay, it cuts it at Right Angles.

For, * find E the Centre of the Circle, and let * 1 *of this.*
EA, EB, be joined.

Then becauſe AF is equal to FB, and FE is common, the two Sides AF, FE, are equal to the two Sides BF, FE, each to each; but the Baſe EA is equal to the Baſe EB. Wherefore the Angle AFE
ſhall be † equal to the Angle BFE. But when a † 8. 1.
Right Line ſtanding upon a Right Line makes the adjacent Angles equal to one another, each of the equal
Angles is ‡ a Right Angle. Wherefore AFE, or ‡ *Def.* 10. 1.
BFE, is a Right Angle. And therefore the Right Line CD drawn thro' the Centre, biſecting the Right Line AB, not drawn thro' the Centre, cuts it at Right Angles. Now, if CD cuts AB at Right Angles, I ſay, it will biſect it; that is, AF will be equal to FB. For the ſame Conſtruction remaining, becauſe EA, being drawn from the Centre, is equal to EB, the
Angle EAF ſhall be * equal to the Angle EBF. But * 5. 1.
the Right Angle AFE is equal to the Right Angle BFE: Therefore the two Triangles EAF, EBF, have two Angles of the one equal to two Angles of the other, and the Side EF is common to both. Where-
fore the other Sides of the one ſhall be † equal to the † 26. 1.
other Sides of the other: And ſo AF will be equal to FB. Therefore, *if in a Circle, a Right Line drawn thro' the Centre cuts any other Right Line, not drawn thro' the Centre, into two equal Parts, it ſhall cut it at Right Angles; and if it cuts it at Right Angles, it ſhall cut it into two equal Parts;* which was to be demonſtrated.

PROPOSITION IV.

THEOREM.

If in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts.

LET A B C D be a Circle, wherein two Right Lines A C, B D, not drawn thro' the Centre, cut each other in the Point E. I ſay, they do not biſect each other.

For, if poſſible, let them biſect each other, ſo that A E be equal to E C, and B E to E D. Let the Centre
† 1 *of this.* F of the Circle A B C D be † found, and join E F.

Then, becauſe the Right Line F E, drawn thro' the Centre, biſects the Right Line A C, not drawn thro'
* 3 *of this.* the Centre, it will * cut A C at Right Angles. And ſo F E A is a Right Angle. Again, becauſe the Right Line F E, drawn thro' the Centre, biſects the Right Line B D, not drawn thro' the Centre, it will * cut B D at Right Angles. Therefore F E B is a Right Angle. But F E A has been ſhewn to be alſo a Right Angle. Wherefore the Angle F E A will be equal to the Angle F E B, a leſs to a greater; which is abſurd. Therefore, A C, B D, do not mutually biſect each other. And ſo, *if in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts*; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

If two Circles cut one another, they ſhall not have the ſame Centre.

LET the two Circles A B C, C D G, cut each other in the Points B, C. I ſay, they have not the ſame Centre.

For, if they have, let it be E, and join E C, and draw E F G at Pleaſure.

Now, becauſe E is the Centre of the Circle A B C; C E will be equal to E F. Again, becauſe E is the Centre of the Circle C D G, C E is equal to the E G. But C E has been ſhewn to be equal to E F. Therefore E F ſhall be equal to E G, a leſs to a greater, which cannot be. Therefore the Point E is not the Centre of both the Circles A B C, C D G. Wherefore, *if two Circles cut one another, they ſhall not have the ſame Centre*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

If two Circles touch one another inwardly, they will not have one and the ſame Centre.

LET two Circles A B C, C D E, touch one another inwardly in the Point C. I ſay, they will not have one and the ſame Centre.

For, if they have, let it be F, and join F C, and draw F B any how.

Then, becauſe F is the Centre of the Circle A B C, C F is equal to F B. And becauſe F is alſo the Centre of the Circle C D E, C F ſhall be equal to F E. But C F has been ſhewn to be equal to F B. Therefore F E is equal to F B, a leſs to a greater; which cannot be. Therefore the Point F is not the Centre of both the Circles A B C, C D E. Wherefore, *if two Circles touch one another inwardly, they will not have one and the ſame Centre*; which was to be demonſtrated.

PROPOSITION VII.

Theorem.

If in the Diameter of a Circle ſome Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greateſt of theſe Lines ſhall be that wherein the Centre of the Circle is, the leaſt, the Remainder of the ſame Line. And of all the other Lines, the neareſt to that which was drawn thro' the Centre, is always greater than that more remote; and only two equal Lines fall from the aboveſaid Point upon the Circumference, on each Side of the leaſt or greateſt Line.

LET A B C D be a Circle, whoſe Diameter is A D, in which aſſume ſome Point F, which is not the Centre of the Circle. Let the Centre of the Circle be E; and from the Point F, let certain Right Lines F B, F C, F G, fall on the Circumference: I ſay, F A is the greateſt of theſe Lines, and F D the leaſt; and of the others F B is greater than F C, and F C greater than F G.

For, let B E, C E, G E, be joined.

* 20. 1. Then, becauſe two Sides of every Triangle are * greater than the third; B E and E F are greater than B F. But A E is equal to B E. Therefore B E and E F are equal to A F. And ſo A F is greater than F B.

† 24. 1. Again, becauſe B E is equal to C E, and F E is common, the two Sides B E and F E are equal to the two Sides C E and E F. But the Angle B E F is greater than the Angle C E F. Wherefore the Baſe B F is greater than the Baſe F C †. For the ſame Reaſon, C F is greater than F G.

‡ 20. 1. Again, becauſe G F and F E ‡ are greater than G E, and G E is equal to E D; G F and F E ſhall be greater than E D; and if F E, which is common, be taken away, then the Remainder G F is greater than the Remainder F D. Wherefore, F A *is the greateſt of the Right Lines, and* F D *the leaſt: Alſo* B F *is greater than* F C, *and* F C *greater than* F G.

I ſay, moreover, that there are only two equal Right Lines, that can fall from the Point F on A B C D, the Circumference of the Circle on each Side the ſhorteſt Line F D. For at the given Point E, with the Right Line E F, make ‡ the Angle F E H equal to the Angle G E F, and join F H. Now becauſe G E is equal to E H, and E F is common, the two Sides G E and E F are equal to the two Sides H E and E F. But the Angle G E F is equal to the Angle H E F. Therefore the Baſe F G ſhall be † equal to the Baſe F H. I ſay, no other Right Line falling from the Point F, on the Circle, can be equal to F G. For if there can, let this be F K. Now, ſince F K is equal to F G, and F H is alſo equal to F G; therefore F K will be equal to F H, *viz.* a Line drawn nigher to that paſſing thro' the Centre, equal to one more remote, which * cannot be. *If*, therefore, *in the Diameter of a Circle, ſome Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greateſt of theſe Lines ſhall be that wherein the Centre of the Circle is; the leaſt, the Remainder of the ſame Line. And of all the other Lines, the neareſt to that which was drawn thro' the Centre, is always greater than that more remote; and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the leaſt or greateſt Line;* which was to be demonſtrated.

‡ 23. 1.
† 4. 1.
* *by this.*

PROPOSITION VIII.

Theorem.

If ſome Point be aſſumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which paſſes thro' the Centre, but the other any how; the greateſt of the Lines which fall upon the concave Part of the Circumference of the Circle, is that paſſing thro' the Centre; and of the others, that which is neareſt to the Line, paſſing thro' the Centre, is greater than that more remote. But the leaſt of the Lines that fall upon the convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the leaſt, is leſs than that which is farther diſtant; and from that Point there can be drawn only two equal Lines, which ſhall fall on the Circumference on each Side the leaſt Line.

LET ABC be a Circle, out of which take any Point D. From this Point let there be drawn certain Right Lines D A, D E, D F, D C, to the Circle, whereof D A paſſes thro' the Centre. I ſay, D A, which paſſes thro' the Centre, is the greateſt of the Lines falling upon A E F C, the concave Circumference of the Circle: Likewiſe D E is greater than D F, and D F greater than D C. But of the Lines that fall upon H L K G the convex Circumference of the Circle, the leaſt is D G, *viz.* the Line drawn from D, to the Diameter G A; and that which is neareſt the leaſt D G, is always leſs than that more remote; that is, D K is leſs than D L, and D L leſs than C H.

* 1 *of this.* For, find * M the Centre of the Circle A B C, and let M E, M F, M C, M H, M L, M K, be joined.

Now, becauſe A M is equal to E M; if M D, which is common, be added, A D will be equal to

† 20. 1. E M and M D. But E M and M D are † greater than

E D; therefore A D is alſo greater than E D. Again,
becauſe M E is equal to M F, and M D is common,
then M E and M D ſhall be equal to M F and M D;
but the Angle E M D is greater than the Angle
F M D. Therefore the Baſe E D will be † greater † 24. 1.
than the Baſe F D. We prove, in the ſame manner,
that F D is greater than C D. Wherefore, D A *is the*
greateſt of the Right Lines falling from the Point D; D E
is greater than D F, *and* D F *is greater than* D C.

Moreover, becauſe M K and K D are * greater * 20. 1.
than M D, and M G is equal to M K; then the Remainder
K D will † be greater than the Remainder † Ax. 4.
G D. And ſo G D is leſs than K D, and conſequently
is the leaſt. And becauſe two Right Lines
M K, K D, are drawn from M and D to the Point
K, within the Triangle M L D, M K, and K D, are
‡ leſs than M L and L D; but M K is equal to M L. ‡ 21. 1.
Wherefore the Remainder D K is leſs than the Remainder
D L. In like manner we demonſtrate, that
D L is leſs than D H. Therefore, D G *is the leaſt*;
and D K *is leſs than* D L, *and* D L *than* D H.

I ſay likewiſe, that from the Point D only two equal
Right Lines can fall upon the Circle on each Side the
leaſt Line. For, make * the Angle D M B at the * 23. 1.
Point M, with the Right Line M D, equal to the Angle
K M D, and join D B. Then, becauſe M K is
equal to M B, and M D is common, the two Sides
K M, M D, are equal to the two Sides M B, M D,
each to each; but the Angle K M D is equal to the
Angle B M D. Therefore the Baſe D K is † equal † 4. 1.
to the Baſe D B. Now I ſay, no other Line can be
drawn from the Point D to the Circle equal to D K;
for, if there can, let it be D N. Now, ſince D K is
equal to D N, as alſo to D B, therefore D B ſhall be
equal to D N, *viz.* the Line drawn neareſt to the leaſt
equal to that more remote, which has been * ſhewn to * by this.
be impoſſible. Therefore, *if ſome Point be aſſumed*
without a Circle, and from it certain Right Lines be drawn
to the Circle, one of which paſſes thro' the Centre, but the
others any how; the greateſt of the Lines, that fall upon the
concave Part of the Circumference of the Circle, is that
paſſing thro' the Centre; and of the others, that which is
neareſt to the Line, paſſing thro' the Centre, is greater than
that more remote. But the leaſt of the Lines that fall upon

the convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the leaſt, is leſs than that which is farther diſtant, and from that Point there can be drawn only two equal Lines, which ſhall fall on the Circumference on each Side the leaſt Line; which was to be demonſtrated.

PROPOSITION IX.

THEOREM.

If a Point be aſſumed in a Circle, and from it more than two equal Right Lines be drawn to the Circumference; then that Point is the Centre of the Circle.

LET the Point D be aſſumed within the Circle A B C; and from the Point D, let there fall more than two equal Right Lines to the Circumference, *viz.* the Right Lines D A, D B, D C. I ſay, the aſſumed Point D is the Centre of the Circle A B C.

For, if it be not, let E be the Centre, if poſſible; and join D E, which produce to G and F.

Then F G is a Diameter of the Circle A B C; and ſo, becauſe the Point D, not being the Centre of the Circle, is aſſumed in the Diameter F G; therefore
* *7 of this.* D G will * be the greateſt Line drawn from D to the Circumference, and D C greater than D B, and D B than D A; but they are alſo equal, which is abſurd. Therefore E is not the Centre of the Circle A B C. And in this manner we prove, *that no other Point, except* D, *is the Centre; therefore* D *is the Centre of the Circle* A B C; which was to be demonſtrated.

Otherwiſe:

Let A B C be the Circle, within which take the Point D, from which let more than two equal Right Lines fall on the Circumference of the Circle, *viz.* the three equal ones D A, D B, D C: I ſay, the Point D is the Centre of the Circle A B C.

† 10. 1. For, join A B, B C, which biſect † in the Points E and Z; as alſo join E D, D Z; which produce to

the Points H, K, O, L; then, becauſe A E is equal to E B, and E D is common, the two Sides A E, E D, ſhall be equal to the two Sides B E, E D. And the Baſe D A, is equal to the Baſe D B: Therefore the Angle A E D will be * equal to the Angle B E D; * 8. 1.
and ſo [by *Def.* 10. 1.] each of the Angles A E D, B E D, is a Right Angle: Therefore H K, biſecting A B, cuts it at Right Angles. And becauſe a Right Line in a Circle, biſecting another Right Line, cuts it at Right Angles, and the Centre of the Circle is in the cutting Line, [by *Cor.* 1. 3.] therefore the Centre of the Circle A B C will be in H K. For the ſame Reaſon, the Centre of the Circle will be in O L. And the Right Lines H K, O L, have no other Point common but D: *Therefore* D *is the Centre of the Circle* A B C; which was to be demonſtrated.

PROPOSITION X.

THEOREM.

A Circle cannot cut another Circle in more than two Points.

FOR, if it can, let the Circle A B C cut the Circle D E F in more than two Points, *viz.* in B, G, F; and let K be the Centre of the Circle A B C, and join K B, K G, K F.

Now, becauſe the Point K is aſſumed within the Circle D E F, from which more than two equal Right Lines K B, K G, K F, fall on the Circumference, the Point K ſhall be † the Centre of the Circle D E F. † 9 *of this.*
But K is ‡ the Centre of the Circle A B C. Therefore ‡ *By Hyp.*
K will be the Centre of two Circles cutting each other; which is * abſurd. Wherefore, *a Circle cannot cut a* * 5 *of this.*
Circle in more than two Points; which was to be demonſtrated.

PROPOSITION XI.

THEOREM.

If two Circles touch each other on the Inſide, and the Centres be found, the Line joining their Centres will fall on the [Point of] Contact of thoſe Circles.

LET two Circles A B C, A D E, touch one another inwardly in A; and let F be the Centre of the Circle A B C, and G that of A D E. I ſay, a Right Line joining the Centres G and F, being produced, will fall in the Point A.

If this be denied, let the Right Line, joining F G, cut the Circles in D and H.

Now, becauſe A G and F G are greater than A F, * that is, than F H; take away F G, which is common, and the Remainder A G is greater than the Remainder G H. But A G is equal to G D; therefore G D is greater than G H, the leſs than the greater; which is abſurd. Whereſore, *a Line drawn thro' the Points* F *and* G, *will not fall out of the Point of Contact* A, *and ſo neceſſarily muſt fall on it*; which was to be demonſtrated.

* 20. 1.

PROPOSITION XII.

THEOREM.

If two Circles touch one another on the Outſide, a Right Line joining their Centres will paſs thro' the [Point of] Contact.

LET two Circles A B C, A D E, touch one another outwardly in the Point A; and let F be the Centre of the Circle A B C, and G that of A D E. I ſay, a Right Line drawn thro' the Centres F and G, will paſs thro' the Point of Contact A.

For, if it does not, let, if poſſible, F C D G, fall without it, and join F A, A G.

Now, ſince F is the Centre of the Circle A B C, A F will be equal to F C. And becauſe G is the

Centre of the Circle A D E, A G will be equal to G D: But A F has been ſhewn to be equal to F C; therefore F A and A G are equal to F C and D G. And ſo the whole F G is greater than F A and A G; and alſo leſs, * which is abſurd. Therefore, *a Right* * 20. 1. *Line, drawn from the Point* F *to* G, *will paſs thro' the Point of Contact* A; which was to be demonſtrated.

PROPOSITION XIII.

Theorem.

One Circle cannot touch another in more Points than one, whether it be inwardly or outwardly.

FOR, in the firſt Place, if this be denied, let the Circle A B D C, if poſſible, touch the Circle E B F D inwardly, in more Points than one, *viz.* in B, and D.

And let G be the Centre of the Circle A B D C, and H that of E B F D.

Then a Right Line, drawn from the Point G to H, will † fall in the Points B and D. Let this Line be † 11 of this. B G H D. And becauſe G is the Centre of the Circle A B D C, the Line B G will be equal to G D. Therefore B G is greater than H D, and B H much greater than H D. Again, ſince H is the Centre of the Circle E B F D, the Line B H is equal to H D. But it has been proved to be much greater than it, which is abſurd. Therefore, *one Circle cannot touch another Circle inwardly in more Points than one.*

Secondly, Let the Circle A C K, if poſſible, touch the Circle A B D C outwardly, in more Points than one, *viz.* in A and C; and let A and C be joined.

Now, becauſe two Points, A and C, are aſſumed in the Circumference of each of the Circles A B D C, A C K, a Right Line joining theſe two Points will fall ‡ within either of the Circles. But it falls within the ‡ 2 of this. Circle A B D C, and without the Circle A C K, which is abſurd. Therefore one Circle cannot touch another Circle in more Points than one outwardly. But it has been proved, that one Circle cannot touch another Circle inwardly [in more Points than one]. Wherefore, *one Circle cannot touch another in more Points than one,*

one, whether it be inwardly or outwardly; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

Equal Right Lines in a Circle are equally diſtant from the Centre; and Right Lines, which are equally diſtant from the Centre, are equal between themſelves.

LET A B D C be a Circle, wherein are the equal Right Lines A B, C D. I ſay, theſe Lines are equally diſtant from the Centre of the Circle.

For, let E be the Centre of the Circle A B D C; from which let there be drawn E F and E G, perpendicular to A B and C D; and let A E and E C be joined.

Then, becauſe a Right Line E F, drawn thro' the Centre, cuts the Right Line A B, not drawn thro' the Centre, at Right Angles, it will * biſect the ſame. Wherefore A F is equal to F B, and ſo A B is double to A F. For the ſame Reaſon C D is double to C G; but A B is equal to C D; therefore A F is equal to C G: And becauſe A E is equal to E C, the Square of A E ſhall be equal to the Square of E C. But the Squares of A F and F E are † equal to the Square of A E; for the Angle at F is a Right Angle: And the Squares of E G and G C are equal to the Square of E C, ſince the Angle at G is a Right one. Therefore the Squares of A F and F E are equal to the Squares of C G and G E. But the Square of A F is equal to the Square of C G; for A F is equal to C G. Therefore the Square of F E is equal to the Square of E G; and ſo F E equal to E G. Alſo Lines in a Circle are ‡ ſaid to be equally diſtant from the Centre, when Perpendiculars drawn to them from the Centre are equal. Therefore, A B *and* C D *are equally diſtant from the Centre.*

* 3 *of this.*

† 47. 1.

‡ *Def.* 4 *of this.*

But if A B and C D are equally diſtant from the Centre, that is, if F E be equal to E G, I ſay, A B is equal to C D.

For, the ſame Conſtruction being ſuppoſed, we demonſtrate, as above, that A B is double to A F, and

CD to CG; and becauſe AE is equal to EC, the Square of AE will be equal to the Square of EC. But the Squares of EF and FA are † equal to the Square † 47. 1. of AE; alſo the Squares of EG and GC are equal †, to the Square of EC. Therefore the Squares of EF and FA are equal to the Squares of EG and GC. But the Square of EG is equal to the Square of EF; for EG is equal to EF. Therefore the Square of AF is equal to the Square of CG; and ſo AF is equal to CG. But AB is double to AF, and CD to CG; whence AB is equal to CD. Therefore, *equal Right Lines in a Circle are equally diſtant from the Centre; and Right Lines, which are equally diſtant from the Centre, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

A Diameter is the greateſt Line in a Circle; and of all the other Lines therein, that which is neareſt to the Centre is greater than that more remote.

LET ABCD be a Circle whoſe Diameter is AD, and Centre E; and let BC be nearer to the Centre than FG. I ſay, AD is the greateſt, and BC is greater than FG.

For, let the Perpendiculars EH, EK, be drawn from the Centre E to BC, FG. Now, becauſe BC is nearer to the Centre than FG, EK will be greater than EH. Let EL be equal to EH; draw LM thro' L at Right Angles to EK, which produce to N; and let EM, EN, EF, EG, be joined.

Then, becauſe EH is equal to EL, the Line BC will be equal to MN*. And, ſince AE is equal to * 14 of this. EM, and DE to EN, AD will be equal to ME and EN. But ME and EN are † greater than MN: † 20. 1. And ſo AD is greater than MN; and NM is equal to BC. Therefore AD is greater than BC. And ſince the two Sides EM, EN, are equal to the two Sides FE, EG, and the Angle MEN greater than the Angle FEG, the Baſe MN ſhall be ‡ greater ‡ 24. 1. than the Baſe FG. But MN is equal to BC.

Therefore B C is greater than F G. And ſo the Diameter A D is the greateſt, and B C is greater than F G. Wherefore, *the Diameter is the greateſt Line in a Circle; and of all the other Lines therein, that which is neareſt to the Centre is greater than that more remote;* which was to be demonſtrated.

PROPOSITION XVI.

THEOREM.

A Line drawn from the extreme [Point] of the Diameter of a Circle, at Right Angles to that Diameter, ſhall fall without the Circle; and between the ſaid Right Line, and the Circumference, no other Right Line can be drawn; and the Angle of a Semicircle is greater than any Right-lined acute Angle; and the remaining Angle [viz. *without the Circumference*] *is leſs than any Right-lined Angle.*

LET A B C be a Circle, whoſe Centre is D, and Diameter A B. I ſay, a Right Line, drawn from the Point A at Right Angles to A B, falls without the Circle.

For, if it does not, let it fall, if poſſible, within the Circle, as A C; and join D C.

Now, becauſe D A is equal to D C, the Angle D A C ſhall be * equal to the Angle A C D. But D A C is a Right Angle; therefore A C D is a Right Angle: And accordingly the Angles D A C, A C D, are equal to two Right Angles; which is abſurd †. Therefore a Right Line, drawn from the Point A at Right Angles to B A, will not fall within the Circle; and ſo likewiſe we prove, that it neither falls in the Circumference. Therefore, *it will neceſſarily fall without the ſame*; which now let be A E.

* 5. 1.

† 17. 1.

Again, between the Right Line A E and the Circumference C H A, no other Right Line can be drawn. For, if there can, let it be F A, and let ‡ D G be drawn, from the Centre D, at Right Angles to F A.

‡ 12. 1.

Now, becauſe A G D is a Right Angle, and D A G is leſs than a Right Angle, D A will be greater than D G *. But D A is equal to D H. Therefore D H is greater

* 19. 1.

greater than D G, the leſs than the greater; which is abſurd. Wherefore, *no Right Line can be drawn between* A E, *and the Circumference* A H C. I ſay, moreover, that the Angle of the Semicircle, contained under the Right Line B A, and the Circumference C H A, is greater than any Right-lined acute Angle; and the remaining Angle contained under the Circumference C H A, and the Right Line A E, is leſs than any Right-lined Angle.

For if any Right-lined acute Angle be greater than the Angle contained under the Right Line B A, and the Circumference C H A; or if any Right-lined Angle be leſs than that contained under the Circumference C H A, and the Right Line A E; then a Right Line may be drawn between the Circumference C H A, and the Right Line A E, making an Angle (contained under Right Lines) greater than that contained under the Right Line B A, and the Circumference C H A, and leſs than that contained under the Circumference C H A, and the Right Line A E. But ſuch a Right Line cannot be drawn, from what has been proved. Therefore, *no Right-lined acute Angle is greater than the Angle contained under the Right Line* B A, *and the Circumference* C H A; *nor leſs than the Angle contained under the Circumference* C H A, *and the Right Line* A E; which was to be demonſtrated.

Coroll. From hence it is manifeſt, that a Right Line, drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle, and that in one Point only; becauſe, if it ſhould meet it in two Points, it would fall within the ſame; * *as has been demonſtrated.* * *2 of this.*

PROPOSITION XVII.

PROBLEM.

To draw a Right Line from a given Point, that ſhall touch a given Circle.

LET A be the Point given, and B C D the Circle. It is required to draw a Right Line from the Point A, that ſhall touch the given Circle B C D.

Let E be the Centre of the Circle, and join A E;
then about the Centre E, with the Diſtance E A, de-
* 11. 1. ſcribe the Circle A F G; draw D F * at Right An-
gles to E A, and join E B F, and A B. I ſay, the
Right Line A B is drawn from the Point A, touching
the Circle B C D.

For, ſince E is the Centre of the Circles B C D,
A F G, the Line E A will be equal to E F, and E D
to E B. Therefore the two Sides A E, E B, are equal
to the two Sides F E, E D, each to each; and they
contain the common Angle E. Wherefore the Baſe
† 4. 1. D F is † equal to the Baſe A B, and the Triangle
D E F equal to the Triangle E B A, and the remain-
ing Angles of the one equal to the remaining Angles
of the other. And ſo the Angle E B A is equal to the
Angle E D F. But E D F is a Right Angle. Where-
fore E B A is alſo a Right Angle, and E B is a Line
drawn from the Centre; but a Right Line, drawn
from the Extremity of the Diameter of a Circle at
‡ Cor. 16. of Right Angles ‡ to it, touches the Circle. Where-
this. fore, A B *touches the Circle*; which was to be done.

PROPOSITION XVIII.

Theorem.

If any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent.

LET any Right Line D E touch a Circle A B C in the Point C, and let there be drawn the Right Line F C from the Centre F. I ſay, F C is perpendicular to D E.

* 12. 1. For, if it be not, let F G be drawn * from the Cen-
tre F, perpendicular to D E.

Now, becauſe the Angle F G C is a Right Angle,
† Cor. 3. of the Angle G C F will be † an acute Angle; and ac-
32. 1. cordingly the Angle F G C is greater than the Angle
‡ 19. 1. F C G; but the greater Angle ſubtends ‡ the greater
Side. Therefore F C is greater than F G. But F C
is equal to F B. Wherefore F B is greater than F G,
a leſs

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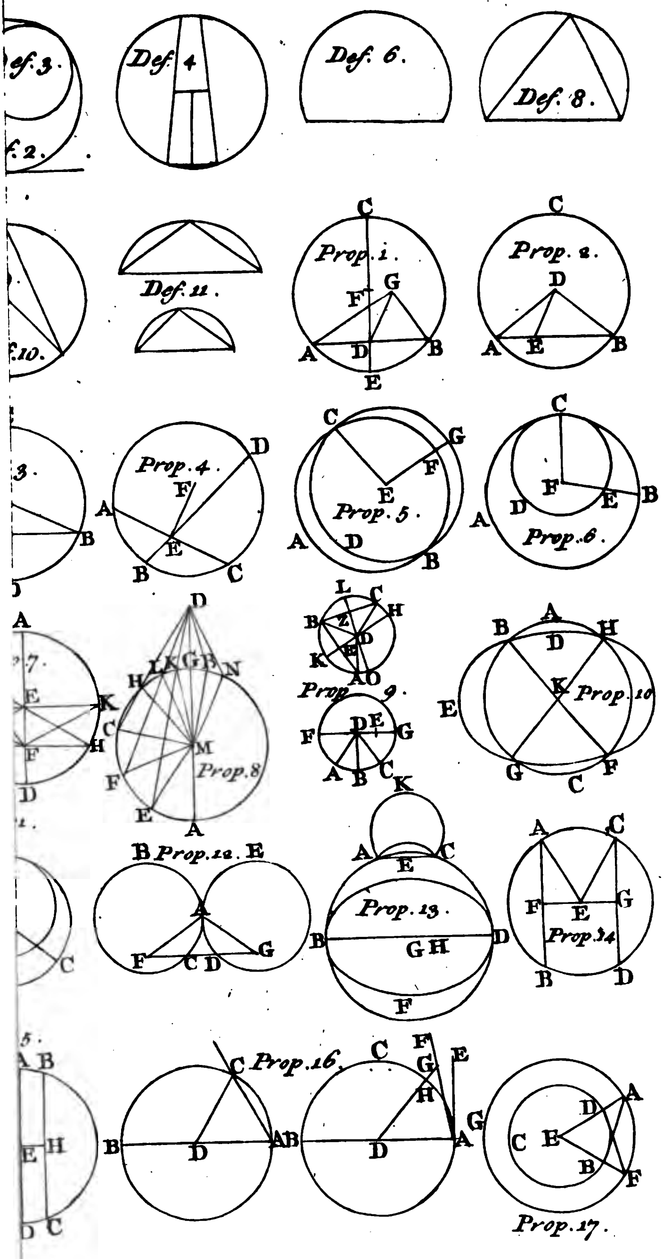

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a leſs than a greater; which is abſurd. Therefore F G is not perpendicular to D E. And in the ſame manner we prove, that no other Right Line but F C is perpendicular to D E. Wherefore F C is perpendicular to D E. Therefore, *if any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn, that Line will be perpendicular to the Tangent*; which was to be demonſtrated.

PROPOSITION XIX.

Theorem.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle ſhall be in the ſaid Line.

LET any Right Line D E touch the Circle A B C in C, and let C A be drawn from the Point C at Right Angles to D E. I ſay, the Circle's Centre is in A C.

For if it be not, let F be the Centre, if poſſible; and join C F.

Then, becauſe the Right Line D E touches the Circle A B C, and F C is drawn from the Centre to the Point of Contact; F C will be perpendicular to D E *. And ſo the Angle F C E is a Right one. But A C E is alſo a Right Angle †: Therefore the Angle F C E is equal to the Angle A C E, a leſs to a greater; which is abſurd. Therefore F is not the Centre of the Circle A B C. After this manner we prove, that the Centre of the Circle can be in no other Line, but A C. Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle ſhall be in the ſaid Line*; which was to be demonſtrated.

* 18 *of this.*

† *From the Hyp.*

PROPOSITION XX.

THEOREM.

The Angle at the Centre of the Circle is double to the Angle at the Circumference, when the ſame Arc is the Baſe of both Angles.

LET ABC be a Circle, at the Centre E whereof is the Angle BEC, and at the Circumference, the Angle BAC, both of which ſtand upon the ſame Arc BC. I ſay, the Angle BEC is double to the Angle BAC.

For join A and produce it to F.

Then, becauſe EA is equal to EB, the Angle
* 5. 1. EAB ſhall be equal to the Angle EBA *. Therefore the Angles EAB, EBA, are double to the An-
† 32. 1. gle EAB; but the Angle BEF is † equal to the Angles EAB, EBA; therefore the Angle BEF is double to the Angle EAB. For the ſame Reaſon, the Angle FEC is double to EAC. Therefore the whole Angle BEC is double to the whole Angle BAC. Again, let there be another Angle BDC; and join DE, which produce to G. We demonſtrate, in the ſame manner, that the Angle GEC is double to the Angle GDC; whereof the Part GEB is double to the Part GDB. And therefore the remaining Part BEC is double to the remaining Part BDC. Conſequently, *an Angle at the Centre of a Circle is double to the Angle at the Circumference, when the ſame Arc is the Baſe of both Angles*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

Angles that are in the ſame Segment of a Circle, are equal to each other.

LET ABCDE be a Circle, and let BAD, BED, be Angles in the ſame Segment thereof BAED. I ſay, thoſe Angles are equal.

For, let F be the Centre of the Circle ABCDE; and join B F, F D.

Now, becauſe the Angle B F D is at the Centre, and the Angle B A D at the Circumference, and they ſtand upon the ſame Arc B C D; the Angle B F D will be ‡ double to the Angle B A D. For the ſame ‡ 20 *of this.* Reaſon, the Angle B F D is alſo double to the Angle B E D. Therefore the Angle B A D will be equal to the Angle B E D.

If the Angles B A D, B E D, are in a Segment leſs than a Semicircle, let A E be drawn; and then all the Angles of the Triangle A B G are † equal to † 32. 1. all the Angles of the Triangle D E G. But the Angles A B E, A D E, are equal, from what has been before proved; and the Angles A G B, D G E, are alſo equal ‡; for they are vertical Angles. Where- ‡ 15. 1. fore the remaining Angle B A G is equal to the remaining Angle G E D. Therefore, *Angles that are in the ſame Segment of a Circle, are equal to each other;* which was to be demonſtrated.

PROPOSITION XXII.

Theorem.

The oppoſite Angles of any quadrilateral Figure, deſcribed in a Circle, are equal to two Right Angles.

LET A B D C be a Circle, wherein is deſcribed the quadrilateral Figure A B C D. I ſay, two oppoſite Angles thereof are equal to two Right Angles.

For join A D, B C.

Then, becauſe the three Angles of any Triangle are * equal to two Right Angles, the three Angles of the * 32. 1. Triangle A B C, *viz.* the Angles C A B, A B C, B C A, are equal to two Right Angles. But the Angle A B C is † equal to the Angle A D C; for they † 21 *of this.* are both in the ſame Segment A B D C. And the Angle A C B is † equal to the Angle A D B, becauſe they are in the ſame Segment A C D B; therefore the whole Angle B D C is equal to the Angles A B C, A C B; and if the common Angle B A C be added, then the Angles B A C, A B C, A C B, are equal to the Angles B A C, B D C; but the Angles B A C,

§ 32. 1. A B C, A C B, are equal § to two Right Angles. Therefore likewiſe, the Angles B A C, B D C, ſhall be equal to two Right Angles. And after the ſame Way we prove, that the Angles A B D, A C D, are alſo equal to two Right Angles. Therefore, *the oppoſite Angles of any quadrilateral Figure, deſcribed in a Circle, are equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XXIII.

THEOREM.

Two ſimilar and unequal Segments of two Circles cannot be ſet upon the ſame Right Line, and on the ſame Side thereof.

FOR if this be poſſible, let the two ſimilar and unequal Segments A C B, A D B, of two Circles, ſtand upon the Right Line A B on the ſame Side thereof. Draw A C D, and let C B, B D, be joined. Now, becauſe the Segment A C B is ſimilar to the Segment A D B, and ſimilar Segments of Circles are * ſuch which include equal Angles; the Angle A C B will be equal to the Angle A D B; the outward one to the inward one; which is † abſurd. Therefore, *ſimilar and unequal Segments of two Circles cannot be ſet upon the ſame Right Line, and on the ſame Side thereof*; which was to be demonſtrated.

* *Def.* 11. *of this.*

† 16. 1.

PROPOSITION XXIV.

THEOREM.

Similar Segments of Circles, being upon equal Right Lines, are equal to one another.

LET A E B, C F D, be ſimilar Segments of Circles, ſtanding upon the equal Right Lines A B, C D. I ſay, the Segment A E B is equal to the Segment C F D.

For the Segment A E B being applied to the Segment C F D, ſo that the Point A coincides with C, and the Line A B with C D; then the Point B will coincide with the Point D, ſince A B and C D are equal,

equal. And ſince the Right Line A B coincides with C D, the Segment A E B will coincide with the Segment C F D. For if, at the ſame Time that A B coincides with C D, the Segment A E B ſhould not coincide with the Segment C F D, but be otherwiſe, as C G D; then a Circle would cut a Circle in more Points than two, *viz.* in the Points C, G, D; which is * impoſſible. Wherefore, if the Right Line A B * 10 *of this.* coincides with C D, the Segment A E B will coincide with, and be equal to, the Segment C F D. Therefore, *ſimilar Segments of Circles, being upon equal Right Lines, are equal to one another*; which was to be demonſtrated.

PROPOSITION XXV.

PROBLEM.

A Segment of a Circle being given, to deſcribe the Circle whereof it is the Segment.

LET A B C be a Segment of a Circle given. It is required to deſcribe a Circle, whereof A B C is a Segment.

Biſect * A C in D, and let D B be drawn † from * 10. 1. the Point D at Right Angles to A C; and join A B. † 11. 1. Now the Angle A B D is either greater, equal, or leſs, than the Angle B A D. And firſt let it be greater, and make ‡ the Angle B A E at the given Point A, ‡ 23. 1. with the Right Line B A, equal to the Angle A B D; produce B D to E, and join E C.

Then, becauſe the Angle A B E is equal to the Angle B A E, the Right Line B E will be * equal to * 6. 1. E A. And becauſe A D is equal to D C, and D E common, the two Sides A D, D E, are each equal to the two Sides C D, D E; and the Angle A D E is equal to the Angle C D E; for each is a Right one. Therefore the Baſe A E is equal to the Baſe E C. But A E has been proved to be equal to E B. Wherefore B E is alſo equal to E C. And accordingly the three Right Lines A E, E B, E C, are equal to each other. Therefore a Circle deſcribed about the Centre E, with either of the Diſtances A E, E B, E C, † ſhall paſs † 9 *of this.* thro' the other Points, and be that required to be deſcribed. But it is manifeſt, that the Segment A B C is leſs than a Semicircle, becauſe the Centre thereof is without the ſame.

But if the Angle A B D be equal to the Angle B A D; then if A D be made equal to B D, or D C, the three Right Lines A D, B D, D C, are equal between themſelves, and D will be the Centre of the Circle to be deſcribed, and the Segment A B C is a Semicircle.

But if the Angle A B D is leſs than the Angle B A D, let the Angle B A E be made, at the given Point A, with the Right Line B A, within the Segment A B C, equal to the Angle A B D.

Then the Point E, in the Right Line D B, will, by arguing as before, appear to be the Centre, and A B C a Segment greater than a Semicircle. Therefore, *a Circle is deſcribed, whereof a Segment is given*; which was to be done.

PROPOSITION XXVI.

THEOREM.

In equal Circles, equal Angles ſtand upon equal Circumferences, whether they be at their Centres, or at their Circumferences.

LET A B C, D E F, be equal Circles; and let B G C, E H F, be equal Angles at their Centres; and B A C, E D F, equal Angles at their Circumferences. I ſay, the Circumference B K C is equal to the Circumference E L F.

For, let B C, E F, be joined. Becauſe A B C,
D E F, are equal Circles, the Lines drawn from their
‡ *Def.* 1. Centres will ‡ be equal. Therefore the two Sides
B G, G C, are equal to the two Sides E H, H F;
and the Angle G is equal to the Angle H. Where-
* 4. 1. fore the Baſe B C is * equal to the Baſe E F. Again,
becauſe the Angle at A is equal to that at D, the
† *Def.* 11. Segment B A C will be † ſimilar to the Segment
E D F; and they are upon equal Right Lines B C,
E F. But thoſe ſimilar Segments of Circles, that are
‡ 24 *of this.* upon equal Right Lines, are ‡ equal to each other.
† *Def.* 11. Therefore the Segment B A C will be † equal to
the Segment E D F. But the whole Circumference ABKCA is equal to the whole Circumference

DELFD. Therefore the remaining Circumference B K C ſhall be equal to the remaining Circumference E L F. Therefore, *in equal Circles, equal Angles ſtand upon equal Circumferences, whether they be at their Centres, or at their Circumferences*; which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

Angles, that ſtand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres, or Circumferences.

LET the Angles B G C, E H F, at the Centres of the equal Circles A B C, D E F, and the Angles B A C, E D F, at their Circumferences, ſtand upon the equal Circumferences B C, E F. I ſay, the Angle B G C is equal to the Angle E H F; and the Angle B A C to the Angle E D F.

For if the Angle B G C be equal to the Angle E H F, it is manifeſt, that the Angle B A C is alſo equal to the Angle E D F: But if the Angle B G C be not equal to the Angle E H F, let one of them be the greater, as B G C, and make * the Angle B G K, at the Point G, with the Line B G, equal to the Angle E H F. * 23. 1. But equal Angles ſtand † upon equal Circumferences, when they are at the Centres. † 26 *of this.* Wherefore the Circumference B K is equal to the Circumference E F. But the Circumference E F is equal to the Circumference B C. Therefore B K is equal to B C, a leſs to a greater, which is abſurd. Wherefore the Angle B G C is not unequal to the Angle E H F, and ſo it muſt be equal to it. But the Angle at A is one half of the Angle B G C; and the Angle at D is one half of the Angle E H F. Therefore the Angle at A is equal to the Angle at D. Wherefore, *Angles, that ſtand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres, or Circumferences*; which was to be demonſtrated.

PRO-

PROPOSITION XXVIII.

THEOREM.

In equal Circles, equal Right Lines cut off equal Parts of the Circumferences; the greater Part of the one Circumference equal to the greater Part of the other, and the leſſer, equal to the leſſer.

LET ABC, DEF, be equal Circles, in which are the equal Right Lines BC, EF, which cut off the greater Circumferences BAC, EDF, and the leſſer Circumferences BGC, EHF. I ſay, the greater Circumference BAC is equal to the greater Circumference EDF, and the leſſer Circumference BGC to the leſſer Circumference EHF.

For aſſume the Centres K and L of the Circles; and join BK, KC, EL, LF.

Becauſe the Circles are equal, the Lines drawn
* Def. 1. from their Centres * are alſo equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and the Baſe BC is equal to the Baſe EF.
† 8. 1. Therefore the Angle BKC is † equal to the Angle
‡ 26 of this. ELF. But equal Angles ſtand ‡ upon equal Circumferences, when they are at the Centres. Wherefore the Circumference BGC is equal to the Circumference EHF, and the whole Circumference ABGCA equal to the whole Circumference DEHFD; and ſo the remaining Circumference BAC ſhall be equal to the remaining Circumference EDF. Therefore, *in equal Circles, equal Right Lines cut off equal Parts of the Circumferences*; which was to be demonſtrated.

PROPOSITION XXIX.

THEOREM.

In equal Circles, the Right Lines, which ſubtend equal Circumferences, are equal.

LET there be two equal Circles, ABC, DEF; and let the equal Circumferences BGC, EHF, be aſſumed in them, and BC, EF, joined. I ſay, the Right Line BC is equal to the Right Line EF.

For, find * K and L, the Centres of the Circles; * 1 *of this.*
and join BK, KC, EL, LF.

Then, becauſe the Circumference BGC is equal
to the Circumference EHF, the Angle BKC ſhall
be † equal to the Angle ELF. And becauſe the Cir- † 27 *of this.*
cles ABGCA, DEHFD, are equal, the Lines drawn
from their Centres ſhall be ‡ equal. Therefore the ‡ *Def.* 1.
two Sides BK, KC, are equal to the two Sides EL,
LF; and they contain equal Angles: Wherefore the
Baſe BC is † equal to the Baſe EF. And ſo, *in* † 4. 1.
equal Circles, the Right Lines, which ſubtend equal Circumferences, are equal; which was to be demonſtrated.

PROPOSITION XXX.

Problem.

To cut a given Circumference into two equal Parts.

LET the given Circumference be ADB. It is required to cut the ſame into two equal Parts.

Join AB, which biſect * in C; and let the Right * 10. 1.
Line CD be drawn from the Point C at Right Angles
to AB †; and join AD, DB. † 11. 1.

Now, becauſe AC is equal to CB, and CD is
common, the two Sides AC, CD, are equal to the
two Sides BC, CD; but the Angle ACD is equal
to the Angle BCD; for each of them is a Right
Angle: Therefore the Baſe AD is † equal to the † 4. 1.
Baſe BD. But equal Right Lines cut ‡ off equal ‡ 28 *of this.*
Circumferences. Wherefore the Circumference AD
ſhall be equal to the Circumference BD. Therefore, *a given Circumference is cut into two equal Parts*;
which was to be done.

PRO-

PROPOSITION XXXI.

THEOREM.

In a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment is leſs than a Right Angle; and the Angle in a leſſer Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a leſſer Segment is leſs than a Right Angle.

LET there be a Circle A B D C, whoſe Diameter is B C, and Centre E; and join B A, A C, A D, D C. I ſay, the Angle which is in the Semicircle B A C is a Right Angle; that which is in the Segment A B C being greater than a Semicircle, *viz.* the Angle A B C is leſs than a Right Angle; and that which is in the Segment A D C being leſs than a Semicircle; that is, the Angle A D C is greater than a Right Angle.

For, join A E, and produce B A to F.

Then, becauſe B E is equal to E A, the Angle
* 5. 1. E A B ſhall be * equal to the Angle E B A. And
* 5. 1. becauſe A E is equal to E C, the Angle A C E will be * equal to the Angle C A E. Therefore the whole Angle B A C, is equal to the two Angles A B C, A C B; but the Angle F A C, being without the Tri-
† 32. 1. angle A B C, is † equal to the two Angles A B C, A C B; therefore the Angle B A C is equal to the
‡ Def. 10, 1. Angle F A C; and ſo each of them is ‡ a Right Angle. Wherefore, *the Angle* B A C, *in a Semicircle, is a Right Angle.* And becauſe the two Angles A B C,
* 17. 1. B A C, of the Triangle A B C *, are leſs than two Right Angles; and B A C is a Right Angle; then, A B C *is leſs than a Right Angle*; *and is, in the Segment* A B C, *greater than a Semicircle.*

And ſince A B C D is a quadrilateral Figure in a Circle, and the oppoſite Angles of any quadrilateral
† 22 of this. Figure deſcribed in a Circle are † equal to two Right Angles; the Angles A B C, A D C, are equal to two Right Angles; and the Angle A B C is leſs than a Right Angle. Therefore, *the remaining Angle* A D C

will be greater than a Right Angle; and is in the Segment A D C, *which is leſs than a Semicircle.*

I ſay, moreover, the Angle of the greater Segment contained under the Circumference A B C and the Right Line A C, is greater than a Right Angle; and the Angle of the leſſer Segment, contained under the Circumference A D C, and the Right Line A C, is leſs than a Right Angle. This manifeſtly appears; for, becauſe the Angle contained under the Right Lines B A, A C, is a Right Angle; the Angle contained under the Circumference A B C, and the Right Line A C, will be greater than a Right Angle. Again, becauſe the Angle contained under the Right Lines C A, A F, is a Right Angle, therefore the Angle which is contained under the Right Line A C, and the Circumference A D C, is leſs than a Right Angle. Therefore, *in a Circle, the Angle that is in the Semicircle is a Right Angle; but the Angle in a greater Segment is leſs than a Right Angle; and the Angle in a leſſer Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a leſſer Segment is leſs than a Right Angle*; which was to be demonſtrated.

PROPOSITION XXXII.

THEOREM.

If any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to thoſe which are made in the alternate Segments of the Circle.

LET any Right Line E F touch the Circle A B C D in the Point B, and let the Right Line B D be any how drawn from the Point B, cutting the Circle. I ſay, the Angles which B D makes with the Tangent Line E F, are equal to thoſe in the alternate Segments of the Circle; that is, the Angle F B D is equal to an Angle made in the Segment D A B, *viz.* to the Angle D A B; and the Angle D B E equal to the Angle D C B, made in the Segment D C B. For,

* 11. 1. Draw * B A, from the Point B, at Right Angles to E F; and take any Point C, in the Circumference B D; and join A D, D C, C B.

Then, becauſe the Right Line E F touches the Circle A B C D in the Point B; and the Right Line B A is drawn from the Point of Contact B, at Right Angles to the Tangent Line; the Centre of the Circle
† 19 of this. A B C D will † be in the Right Line B A; and ſo B A is a Diameter of the Circle, and the Angle
‡ 31 of this. A D B, in a Semicircle is ‡ a Right Angle. There-
* 32. 1. fore the other Angles, B A D, A B D, are * equal to one Right Angle. But the Angle A B F is alſo a Right Angle: Therefore the Angle A B F is equal to the Angles B A D, A B D; and if A B D, which is common, be taken away, then the Angle D B F, remaining, will be equal to that which is in the alternate Segment of the Circle, *viz.* equal to the Angle B A D. And becauſe A B C D is a quadrilateral Figure in a
† 22 of this. Circle, the oppoſite Angles thereof are † equal to two Right Angles; therefore the Angles D B F, D B E, will be equal to the Angles B A D, B C D. But B A D has been proved to be equal to D B F; therefore the Angle D B E is equal to the Angle made in D C B, the alternate Segment of the Circle, *viz.* equal to the Angle D C B. Therefore, *if any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to thoſe which are made in the alternate Segments of the Circle*; which was to be demonſtrated.

PROPOSITION XXXIII.

PROBLEM.

To deſcribe upon a given Right Line, a Segment of a Circle, which ſhall contain an Angle, equal to a given Right-lined Angle.

LET the given Right Line be A B, and C the given Right-lined Angle. It is required to deſcribe the Segment of a Circle upon the given Right Line A B, containing an Angle, equal to the Angle C.

At the Point A, with the Right Line A B, make ‡ the Angle B A D equal to the Angle C, and draw * A E from the Point A, at Right Angles to A D. Likewiſe biſect † A B in F, and let F G be drawn from the Point F, at Right Angles to A B; and join G B.

‡ 23. 1.
* 11. 1.
† 10. 1.

Then, becauſe A F is equal to F B, and F G is common, the two Sides A F, F G, are equal to the two Sides B F, F G; and the Angle A F G is equal to the Angle B F G. Therefore the Baſe A G is ‡ equal to the Baſe G B. And ſo, if a Circle be deſcribed about the Centre G, with the Diſtance A G, this ſhall paſs thro' the Point B. Deſcribe the Circle, which let be A B E, and join E B. Now, becauſe A D is drawn from the Point A, the Extremity of the Diameter A E, at Right Angles to A E, the ſaid A D will * touch the Circle. And ſince the Right Line A D touches the Circle A B E, and the Right Line A B is drawn in the Circle from the Point of Contact A, the Angle D A B is † equal to the Angle made in the alternate Segment, *viz.* equal to the Angle A E B. But the Angle D A B is equal to the Angle C. Therefore the Angle C will be equal to the Angle A E B. Wherefore, *the Segment of a Circle* A E B *is deſcribed upon the given Right Line* A B, *containing an Angle* A E B, *equal to a given Angle* C; which was to be done.

‡ 4. 1.
* Cor. 16. of this.
† 32 of this.

PROPOSITION XXXIV.

PROBLEM.

To cut off a Segment from a given Circle, that ſhall contain an Angle, equal to a given Right-lined Angle.

LET the given Circle be A B C, and the Right-lined Angle given D. It is required to cut off a Segment from the Circle A B C, containing an Angle equal to the Angle D.

Draw ‡ the Right Line E F, touching the Circle in the Point B, and make * the Angle F B C, at the Point B, equal to the Angle D.

‡ 17 of this.
* 23. 1.

Then, becauſe the Right Line E F touches the Circle A B C in the Point B, and B C is drawn from the

* 32 *of this.* the Point of Contact B; the Angle F B C will be * equal to that in the alternate Segment of the Circle; but the Angle F B C is equal to the Angle D. Therefore the Angle in the Segment B A C will be equal to the Angle D. Therefore, *the Segment* B A C *is cut off from the given Circle* A B C, *containing an Angle equal to the given Right-lined Angle* D; which was to be done.

PROPOSITION XXXV.

Theorem.

If two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one is equal to the Rectangle under the Segments of the other.

IN the Circle A B C D, let two Right Lines mutually cut each other in the Point E I ſay, the Rectangle contained under A E and E C is equal to the Rectangle contained under D E and E B.

If A C and D B paſs through the Centre, ſo that E be the Centre of the Circle A B C D; it is manifeſt, ſince A E, E C, D E, E B, are equal, that the Rectangle under A E and E C is equal to the Rectangle under D E and E B.

But if A C, D B, do not paſs through the Centre, aſſume the Centre of the Circle F; from which draw F G, F H, perpendicular to the Right Lines A C, D B; and join F B, F C, F E.

Then, becauſe the Right Line G F, drawn through the Centre, cuts the Right Line A C, not drawn thro' * 3 *of this.* the Centre, at Right Angles, it will alſo biſect * the ſame. Whereſore A G is equal to G C: And becauſe the Right Line A C is cut into two equal Parts in the Point G, and into two unequal Parts in E, the Rectangle under A E and E C, together with the † 5. 2. Square of E G, is † equal to the Square of G C. And if the common Square of G F be added, then the Rectangle under A E and E C, together with the Squares of E G and G F, is equal to the Squares of C G and ‡ 47. 1. G F. But the Square of F E is ‡ equal to the Squares of E G and G F, and the Square of F C equal ‡ to the

Squares of C G and G F. Therefore the Rectangle under A E and E C, together with the Square of F E, is equal to the Square of F C; but C F is equal to F B. Therefore the Rectangle under A E and E C, together with the Square of E F, is equal to the Square of F B. For the ſame Reaſon, the Rectangle under D E and E B, together with the Square of F E, is equal to the Square of F B. But it has been proved, that the Rectangle under A E and E C, together with the Square of F E, is alſo equal to the Square of F B. Therefore the Rectangle under A E and E C, together with the Square of F E, is equal to the Rectangle under D E and E B, together with the Square of F E. And if the common Square of F E be taken away, then there will remain the Rectangle under A E and E C, equal to the Rectangle under D E and E B. Wherefore, *if two Right Lines in a Circle mutually cut each other, the Rectangle, contained under the Segments of the one, is equal to the Rectangle, under the Segments of the other*; which was to be demonſtrated.

PROPOSITION XXXVI.

THEOREM.

If ſome Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its Part between the Convexity of the Circle and the aſſumed Point, will be equal to the Square of the Tangent Line.

LET any Point D be aſſumed without the Circle A B C, and let two Right Lines D C A, D B, fall from the ſaid Point to the Circle; whereof D C A cuts the Circle, and D B touches it. I ſay, the Rectangle under A D and D C is equal to the Square of D B.

Now D C A either paſſes thro' the Centre, or not. In the firſt Place, let it paſs thro' the Centre of the Circle A B C, which let be E, and join E B. Then the Angle E B D is * a Right Angle. And ſo, ſince * 18 *of this.*
the Right Line A C is biſected in E, and C D is added

thereto, the Rectangle under A D and D C, together
* 6. 2. with the Square of E C, ſhall * be equal to the Square
of E D. But E C is equal to E B; wherefore the Rectangle
under A D and D C, together with the Square
of E B, is equal to the Square of E D. But the Square
† 47. 1. of E D is † equal to the Squares of E B and B D, for
the Angle E B D is a Right Angle: Therefore the
Rectangle under A D and D C, together with the
Square of E B, is equal to the Squares of E B and B D;
and if the common Square of E B be taken away, the
Rectangle under A D and D C remaining, will be
equal to the Square of the Tangent Line B D.

Now, let D C A not paſs thro' the Centre of the
‡ 1 *of this.* Circle A B C; and find ‡ the Centre E thereof, and
draw E F perpendicular to A C, and join E B, E C,
E D. Therefore E F D is a Right Angle. And because
a Right Line E F, drawn thro' the Centre, cuts
a Right Line A C, not drawn thro' the Centre, at Right
* 3 *of this.* Angles, it will * biſect the ſame; and ſo A F is equal
to F C. Again, ſince the Right Line A C is biſected
in F, and C D is added thereto, the Rectangle under
A D and D C, together with the Square of F C, will
* 6. 2. be * equal to the Square of F D. And if the common
Square of E F be added, then the Rectangle
under A D and D C, together with the Squares of F C
and F E, is equal to the Squares of D F and F E. But
the Square of D E is equal to the Squares of D F and
F E; for the Angle E F D is a Right one; and the
† 47. 1. Square of C E is † equal to the Squares of C F and
F E. Therefore the Rectangle under A D and D C,
together with the Square of C E, is equal to the Square
of E D; but C E is equal to E B. Wherefore the
Rectangle under A D and D C, together with the
Square of E B, is equal to the Square of E D. But
the Squares of E B and B D are † equal to the Square
of E D; ſince the Angle E B D is a Right one.
Wherefore the Rectangle under A D and D C, together
with the Square of E B, is equal to the Squares of
E B and B D. And if the common Square of E B be
taken away, the Rectangle under A D and D C, remaining,
will be equal to the Square of D B. Therefore,
*if any Point be taken without a Circle, and from that
Point two Right Lines fall to the Circle, one of which cuts
the Circle, and the other touches it; the Rectangle contained*

under the whole Secant Line, and its Part between the Convexity of the Circle and the aſſumed Part, will be equal to the Square of the Tangent Line; which was to be demonſtrated.

PROPOSITION XXXVII.

Theorem.

If ſome Point be taken without a Circle, and two Right Lines be drawn from it to the Circle, ſo that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this laſt Line will touch the Circle.

LET ſome Point D be aſſumed without the Circle A B C, and from it draw two Right Lines D C A, D B, to the Circle, in ſuch manner, that D C A cuts the Circle, and D B falls upon it: And let the Rectangle under A D and D C be equal to the Square of D B. I ſay, the Right Line D B touches the Circle.

For, let the Right Line D E be drawn * touching the Circle A B C, and find F the Centre ‡ of the Circle; and join E F, F B, F D. * 17 of this. ‡ 1 of this.

Then the Angle F E D is † a Right Angle. And because D E touches the Circle A B C, and D C A cuts it, the Rectangle under A D and D C will be equal to the Square of D E. But the Rectangle under A D and D C is ‡ equal to the Square of D B. Wherefore the Square of D E ſhall be equal to the Square of D B. And ſo the Line D E will be equal to the Line D B. But E F is equal to F B: Therefore the two Sides D E, E F, are equal to the two Sides D B, B F; and the Baſe F D is common. Wherefore the Angle D E F is equal * to the Angle D B F: But D E F is a Right Angle; wherefore D B F is alſo a Right Angle, and F B produced is a Diameter. But a Right Line drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle; therefore B D neceſſarily touches the Circle. We prove this in the ſame manner, if the Centre of the Circle be † 18 of this. ‡ By Hyp. * 8. 1.

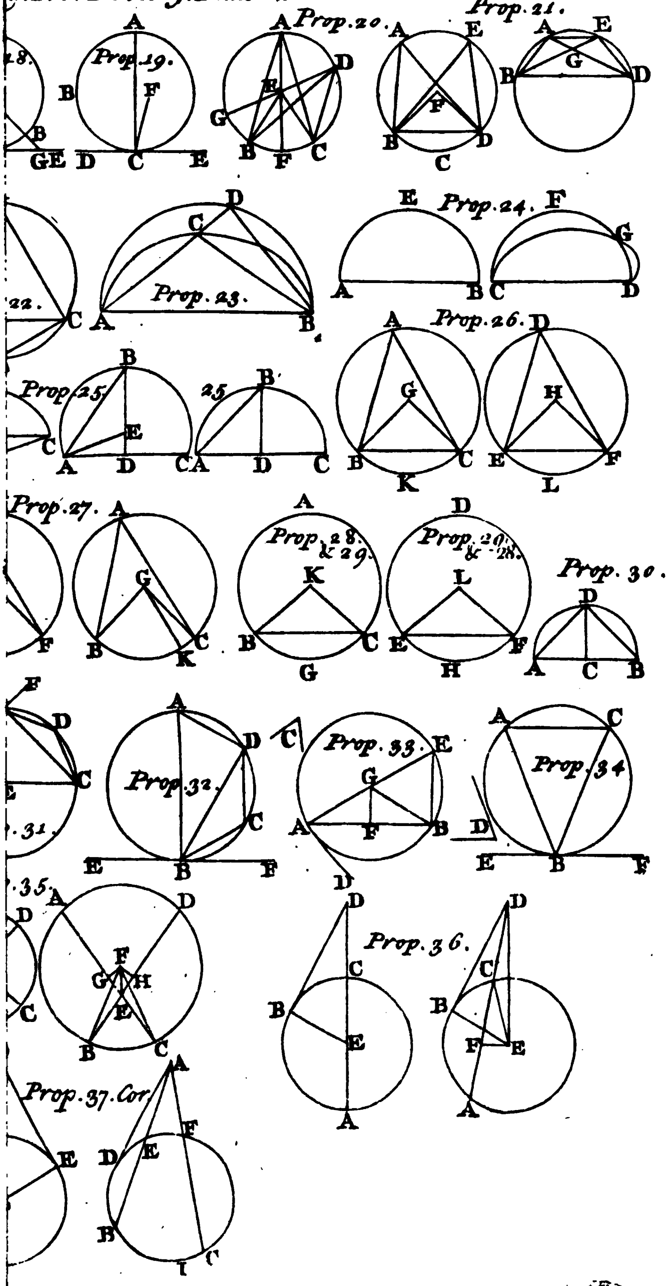
in the Right Line C A. *If*, therefore, *any Point be aſſumed without a Circle, and two Right Lines be drawn from it to the Circle, ſo that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this laſt Line will touch the Circle;* which was to be demonſtrated.

Coroll. Hence, if from any Point, without a Circle, ſeveral Right Lines A B, A C, are drawn, cutting the Circle, the Rectangles comprehended under the whole Lines A B, A C, and their external Parts A E, A F, are equal between themſelves. For, if the Tangent A D be drawn, the Rectangle under B A and A E is equal to the Square of A D; and the Rectangle under C A and A F is equal to the ſame Square of A D: Therefore the Rectangles ſhall be equal.

The END *of the* THIRD BOOK.

EUCLID's

g.100. Book 3. Plate 2.



EUCLID's

ELEMENTS.

BOOK IV.

DEFINITIONS.

I. *A Right-lined Figure is ſaid to be inſcribed in a Right-lined Figure, when every one of the Angles of the inſcribed Figure touches every one of the Sides of the Figure wherein it is inſcribed.*

II. *In like Manner a Figure is ſaid to be deſcribed about a Figure, when every one of the Sides of the Figure, circumſcribed, touches every one of the Angles of the Figure, about which it is circumſcribed.*

III. *A Right-lined Figure is ſaid to be inſcribed in a Circle, when every one of the Angles of that Figure which is inſcribed, touches the Circumference of the Circle.*

IV. *A Right-lined Figure is ſaid to be deſcribed about a Circle, when every one of the Sides of the circumſcribed Figure touches the Circumference of the Circle.*

V. *So likewiſe a Circle is ſaid to be inſcribed in a Right-lined Figure, when the Circumference of* *the Circle touches all the Sides of the Figure in which it is inſcribed.*

VI. *A Circle is ſaid to be deſcribed about a Figure, when the Circumference of the Circle touches all the Angles of the Figure which it circumſcribes.*

VII. *A Right Line is ſaid to be applied in a Circle, when its Extremes are in the Circumference of the Circle.*

PROPOSITION I.

PROBLEM.

To apply a Right Line in a given Circle, equal to a given Right Line, whoſe Length does not exceed the Diameter of the Circle.

LET the Circle given be A B C, and the given Right Line, not greater than the Diameter, be D. It is required to apply a Right Line in the Circle A B C, equal to the Right Line D.

Draw B C the Diameter of the Circle; then, if B C be equal to D, what was required is done: For in the Circle A B C there is applied the Right Line B C, equal to the Right Line D: But if not, the Diameter B C is greater than D, and put * C E equal to D; and about the Centre C, with the Diſtance C E, let the Circle A E F be deſcribed; and join C A. * 3. 1.

Then, becauſe the Point C is the Centre of the Circle A E F, C A will be equal to C E; but D is equal to C E. Wherefore C A is equal to D. And ſo, *in the Circle* A B C, *there is applied a Right Line* C A, *equal to the given Right Line* D, *not greater than the Diameter*; which was to be done.

PRO-

PROPOSITION II.

Problem.

In a given Circle to deſcribe a Triangle equiangular to a given Triangle.

LET A B C be a Circle given, and D E F a given Triangle. It is required to deſcribe a Triangle
in the Circle A B C, equiangular to the Triangle
D E F. Draw the Right Line G A H touching * the * 17. 3.
Circle A B C in the Point A; and with the Right
Line A H, at the Point A, make † an Angle H A C, † 23. 1.
equal to the Angle D E F. Likewiſe, at the ſame
Point A, with the Line A G, make the Angle G A B
equal to the Angle D F E; and join B C.

Then, becauſe the Right Line H A G touches the
Circle A B C, and A C is drawn from the Point of
Contact in the Circle, the Angle H A C ſhall be ‡ ‡ 32. 3.
equal to A B C, the Angle in the alternate Segment of
the Circle. But the Angle H A C is equal to the Angle D E F; therefore alſo the Angle A B C is equal to
the Angle D E F. For the ſame Reaſon, the Angle
A C B is likewiſe equal to the Angle D F E. Wherefore the other Angle B A C ſhall be † equal to the † Cor. 2.
other Angle E D F. And, conſequently, *the Triangle* 32. 1.
A B C *is equiangular to the Triangle* D E F, *and is deſcribed in the Circle* A B C; which was to be done.

PROPOSITION III.

Problem.

About a given Circle to deſcribe a Triangle, equiangular to a Triangle given.

LET A B C be the given Circle, and D E F the given Triangle. It is required to deſcribe a Triangle about the Circle A B C, equiangular to the Triangle D E F.

Produce the Side E F, both Ways, to the Points G and H, and find the Centre of the Circle K, and any how draw the Line K B. Then at the Point K,

* 23. 1. with KB make * the Angle BKA equal to the Angle
DEG; and the Angle BKC, at the ſame Point K
on the other Side the Line KB, equal to the Angle
‡ 17. 3. DFH; and thro' the Points A, B, C, ‡ let the Right
Lines LAM, MBN, NCL, be drawn, touching
the Circle ABC.

Then, becauſe the Lines LM, MN, NL, touch
the Circle ABC in the Points A, B, C, and the Lines
KA, KB, KC, are drawn from the Centre K to the
Points A, B, C; the Angles at the Points A, B, C,
† 18. 3. will be † Right Angles. And becauſe the four An-
gles of the quadrilateral Figure AMBK are equal to
four Right Angles (for it may be divided into two
Triangles), and the Angles KAM, KBM, are
each Right Angles; therefore the other Angles
AKB, AMB, are equal to two Right Angles. But
DEG, DEF, are equal to two Right Angles;
therefore the Angles AKB, AMB, are equal to the
Angles DEG, DEF, whereof AKB is equal to
DEG. Wherefore the other Angle AMB is equal
to the other Angle DEF. In like manner we de-
monſtrate, that the Angle LNB is equal to the An-
‡ Cor. 2. gle DFE. Therefore the other Angle MLN is ‡
32. 1. equal to the other Angle EDF. Wherefore, *the
Triangle* LNM *is equiangular to the Triangle* DEF,
and is deſcribed about the Circle ABC; which was to
be done.

PROPOSITION IV.

PROBLEM.

To inſcribe a Circle in a given Triangle.

LET ABC be a Triangle given. It is required to inſcribe a Circle in the ſame.

* 9. 1. Cut * the Angles ABC, BCA, into two equal
Parts by the Right Lines BD, DC, meeting each
other in the Point D; and from this Point draw DE,
† 12. 1. DF, DG, † perpendicular to the Sides AB, BC, AC.

Now, becauſe the Angle EBD is equal to the Angle FBD, and the Right Angle BED is equal to the Right Angle BFD; then the two Triangles EBD, DBF, have two Angles of the one, equal to two Angles of the other, and one Side DB common to both, *viz.* that which ſubtends the equal Angles; therefore the other Sides of the one Triangle ſhall be * equal to * 26. 1. the other Sides of the other; and ſo D E ſhall be equal to D F. And, for the ſame Reaſon, D G is equal to D F; therefore D E is alſo equal to D G: And ſo the three Right Lines D E, D F, D G, are equal between themſelves. Wherefore a Circle deſcribed about the Centre D, with either of the Diſtances D E, D F, D G, will alſo paſs thro' the other Points. And the Sides A B, B C, A C, will touch it; becauſe the Angles at E, F and G, are Right Angles. For if it ſhould cut them, a Right Line, drawn on the Extremity of the Diameter of a Circle at Right Angles, will fall within the Circle; which is * abſurd. * 16. 1. Therefore a Circle deſcribed about the Centre D, with either of the Diſtances D E, D F, D G; will not cut the Sides A B, B C, C A; wherefore it will touch them, and will be a Circle deſcribed in the Triangle A B C. Therefore, *the Circle* E F G *is deſcribed in the given Triangle* A B C; which was to be done.

PROPOSITION V.

Problem.

To deſcribe a Circle about a given Triangle.

LET A B C be a given Triangle. It is required to deſcribe a Circle about the ſame.

Biſect * the Sides A B, A C, in the Points D, E; * 10. 1. from which Points let D F, E F, be drawn † at Right † 11. 1. Angles to A B, A C, which will meet either within the Triangle A B C, or in the Side B C, or without the Triangle.

Firſt, Let them meet in the Point F within the Triangle; and join B F, F C, F A. Then, becauſe A D is equal to D B, and D F is common, and at Right Angles to A B; the Baſe A F will be ‡ equal ‡ 4. 1. to the Baſe F B. And after the ſame manner we prove, that the Baſe C F is equal to the Baſe F A. Therefore alſo is B F equal to C F: And ſo the three Right Lines F A, F B, F C, are equal to each other. Wherefore, *a Circle deſcribed about the Centre* F, *with either of the Diſtances,* F A, F B, F C, *will paſs alſo* *thro' the other Points, and will be a Circle deſcribed about the Triangle* A B C.

Secondly, Let D F, E F, meet each other in the Point F, in the Side B C, as in the ſecond Figure; and join A F. Then we prove, as before, *that the Point* F *is the Centre of a Circle deſcribed about the Triangle* A B C.

Laſtly, Let the Right Lines D F, E F, meet one another again in the Point F, without the Triangle, as in the third Figure; and join A F, F B, F C. And becauſe A D is equal to D B, and D F is common, and at Right Angles, the Baſe A F ſhall be equal to the Baſe B F. So likewiſe we prove, that C F is alſo equal to A F. Wherefore B F is equal to C F. And ſo again, *if a Circle be deſcribed on the Centre* F, *with either of the Diſtances* F A, F B, F C, *it will paſs thro' the other Points, and will be deſcribed about the Triangle* A B C; which was to be done.

Coroll. If a Triangle be Right-angled, the Centre of the Circle falls in the Side oppoſite to the Right Angle; if acute-angled, it falls within the Triangle; and if obtuſe-angled, it falls without the Triangle.

PROPOSITION VI.

PROBLEM.

To inſcribe a Square in a given Circle.

LET A B C D be a Circle given. It is required to inſcribe a Square within the ſame.

Draw A C, D B, two Diameters of the Circle, † 11. 1. cutting one another at Right Angles †; and join A B, B C, C D, D A.

Then, becauſe B E is equal to E D (for E is the Centre), and E A is common, and at Right Angles * 4. 1. to B D, the Baſe B A ſhall be * equal to the Baſe A D; and for the ſame Reaſon B C, C D, B A, and A D, are all equal to each other. Therefore the quadrilateral Figure A B C D is equilateral. I ſay, it is alſo rectangular. For, becauſe the Right Line D B is a Diameter of the Circle A B C D, therefore

B A D

B A D will be a Semicircle. Wherefore the Angle
B A D is * a Right Angle. And for the ſame Rea- * 31. 3.
ſon, every one of the other Angles A B C, B C D,
C D A, is a Right Angle. Therefore A B C D is a
rectangular quadrilateral Figure: But it has alſo been
proved to be equilateral. Wherefore, *it ſhall neceſſarily
be a Square, and is inſcribed in the Circle* A B C D;
which was to be done.

PROPOSITION VII.

PROBLEM.

To deſcribe a Square about a given Circle.

LET A B C D be a Circle given. It is required
to deſcribe a Square about the ſame.

Draw A C, B D, two Diameters of the Circle,
cutting each other at Right Angles †; and thro' the † 11. 1.
Points A, B, C, D, draw * F G, G H, H K, K F, * 17. 3.
Tangents to the Circle A B C D.

Then, becauſe F G touches the Circle A B C D,
and E A is drawn from the Centre E to the Point of
Contact A, the Angles at A will be † Right Angles. † 18. 3.
For the ſame Reaſon, the Angles at the Points
B, C, D, are Right Angles. And ſince the Angle
A E B is a Right Angle, as alſo E B G, G H ſhall ‡ ‡ 28. 1.
be parallel to A C, and for the ſame Reaſon, A C to
K F. In this manner we prove likewiſe, that G F
and H K are parallel to B E D; and ſo G F is parallel
to H K. Therefore G K, G C, AK, F B, B K,
are Parallelograms; and ſo G F is † equal to H K, † 34. 1.
and G H to F K. And ſince A C is equal to B D,
and A C † equal to either G H or F K; and B D
equal to either G F, or H K; G H, or F K, is equal
to G F, or H K. Therefore F G H K is an equilateral
quadrilateral Figure: I ſay, it is alſo equiangular.
For, becauſe G B E A is a Parallelogram, and
A E B is a Right Angle; then A G B ſhall be alſo a
Right Angle. In like manner we demonſtrate, that
the Angles at the Points H, K, F, are Right Angles.
Therefore the quadrilateral Figure F G H K is rectangular;
but it has been proved to be equilateral likewiſe.
Wherefore, *it muſt neceſſarily be a Square, and
is deſcribed about the Circle* A B C D; which was to
be done.

PROPOSITION VIII.

PROBLEM.

To deſcribe a Circle in a given Square.

LET the given Square be A B C D. It is required to deſcribe a Circle within the ſame.

* 10. 1. Biſect * the Sides A B, A D, in the Points F, E; † 31. 1. and draw † E H thro' E, parallel to A B, or D C; and F K thro' F, parallel † to B C, or A D. Then A K, K B, A H, H D, A G, G C, B G, G D, are ‡ 34. 1. all Parallelograms, and their oppoſite Sides are ‡ equal. And becauſe D A is equal to A B, and A E is half of A D, and A F half of A B, A E ſhall be equal to A F; but the oppoſite Sides are alſo equal. Therefore F G is equal to G E. In like manner we demonſtrate, that G H, or G K, is equal to either F G, or G E. Therefore G E, G F, G H, G K, are equal to each other: And ſo a Circle, being deſcribed about the Centre G, with either of the Diſtances G E, G F, G H, G K, will alſo paſs thro' the other Points, and ſhall touch the Sides D A, A B, B C, C D; becauſe the Angles at E, F, H, K, are Right Angles. For if the Circle ſhould cut the Sides of the Square, a Right Line, drawn from the End of the Diameter of a Circle, at Right Angles, will fall within the Circle; * 16. 3. which is * abſurd. Wherefore a Circle deſcribed about the Centre G, with either of the Diſtances G E, G F, G H, G K, will not cut D A, A B, B C, C D, the Sides of the Square. Wherefore, *it ſhall neceſſarily touch them, and will be deſcribed in the Square* A B C D; which was to be done.

PROPOSITION IX.

PROBLEM.

To deſcribe a Circle about a Square given.

LET A B C D be a Square given. It is required to circumſcribe a Circle about the ſame.

Join A C, B D, mutually cutting one another in the Point E.

And ſince D A is equal to A B, and A C is common, the two Sides D A, A C, are equal to the two Sides B A, A C; but the Baſe D C is equal to the Baſe B C. Therefore the Angle D A C will * be * 8. 1. equal to the Angle B A C: And conſequently the Angle D A B is biſected by the Right Line A C. In the ſame manner we prove, that each of the other Angles A B C, B C D, C D A, are biſected by the Right Lines A C, D B.

Then, becauſe the Angle D A B is equal to the Angle A B C, and the Angle E A B is half of the Angle D A B, and the Angle A E B half of the Angle A B C; the Angle E A B ſhall be equal to the Angle E B A: And ſo the Side E A is † equal to the † 6. 1. Side E B. In like manner we demonſtrate, that each of the Right Lines E C, E D, is equal to each of the Right Lines E A, E B. Therefore the four Right Lines E A, E B, E C, E D, are equal between themſelves. Wherefore, *a Circle being deſcribed about the Centre* E, *with either of the Diſtances* E A, E B, E C, E D, *will alſo paſs thro' the other Points, and will be deſcribed about the Square* A B C D; which was to be done.

PROPOSITION X.

Problem.

To make an Iſoſceles Triangle, having each of the Angles at the Baſe double to the other Angle.

CUT * any given Right Line A B in the Point C, * 11. 2. ſo that the Rectangle contained under A B and B C be equal to the Square of A C; then about the Centre A, with the Diſtance A B, let the Circle B D E be deſcribed; and † in the Circle B D E apply the Right † 1 *of this.* Line B D equal to A C; which is not greater than the Diameter. This being done, join D A, D C, and deſcribe ‡ a Circle A C D about the Triangle A D C. ‡ 5 *of this.*

Then, becauſe the Rectangle under A B and B C is equal to the Square of A C, and A C is equal to B D, the Rectangle under A B and B C ſhall be equal to the Square of B D. And becauſe ſome Point B, is taken without the Circle A C D, and from that Point there fall two Right Lines, B C A, B D, to the Circle,

cle, one of which cuts the Circle; and the other falls
on it; and ſince the Rectangle under AB and BC is
equal to the Square of BD, the Right Line BD ſhall
* 37. 3. * touch the Circle ACD. And ſince BD touches
it, and DC is drawn from the Point of Contact D,
the Angle BDC is equal to the Angle in the alter-
† 32. 3. nate Segment of the Circle, *viz.* equal † to the Angle
DAC. And ſince the Angle BDC is equal to the
Angle DAC; if CDA, which is common, be added,
the whole Angle BDA is equal to the two Angles
‡ 32. 1. CDA, DAC. But the outward Angle BCD is ‡
equal to CDA and DAC. Therefore BDA is
* 5. 1. equal to BCD. But the Angle BDA * is equal to
the Angle CBD, becauſe the Side AD is equal to
the Side AB. Wherefore DBA ſhall be equal to
BCD: And ſo the three Angles BDA, DBA,
BCD, are equal to each other. And ſince the An-
gle DBC is equal to the Angle BCD, the Side BD
† 6. 1. is † equal to the Side DC. But BD is put equal to
CA. Therefore CA is equal to CD. And ſo the
‡ 5. 1. Angle CDA is ‡ equal to the Angle DAC. There-
fore the Angles CDA, DAC, taken together, are
double to the Angle DAC. But the Angle BCD
is equal to the Angles CDA, and DAC. There-
fore the Angle BCD is double to the Angle DAC.
But BCD is equal to BDA, or DBA. Wherefore
BDA, or DBA, is double to DAB. Therefore,
the Iſoſceles Triangle ABD *is made, having each of the
Angles at the Baſe double to the other Angle*; which was
to be done.

PROPOSITION XI.

Problem.

To deſcribe an equilateral and equiangular Pentagon in a given Circle.

LET ABCDE be a Circle given. It is required
to deſcribe an equilateral and equiangular Penta-
gon in the ſame.
* 10 *of this.* Make an Iſoſceles Triangle FGH, having * each
of the Angles at the Baſe GH, double to the other
Angle F; and deſcribe the Triangle ADC in the
† 2 *of this.* Circle ABCDE, equiangular † to the Triangle
FGH;

F G H; ſo that the Angle C A D be equal to that at F, and A C D, C D A, each equal to the Angles G or H. Whereſore the Angles A C D, C D A, are each double to the Angle C A D. This being done, biſect the Angles * A C D, C D A, by the Right Lines* 9. 1. C E, D B, and join A B, B C, D E, E A.

Then, becauſe each of the Angles A C D, C D A, is double to C A D, and they are biſected by the Right Lines C E, D B; the five Angles D A C, A C E, E C D, C D B, B D A, are equal to each other. But equal Angles ſtand † upon equal Circumferences. † 26. 3. Therefore the five Circumferences A B, B C, C D, D E, E A, are equal to each other. But equal Circumferences ſubtend ‡ equal Right Lines. ‡ 29. 3. Therefore the five Right Lines A B, B C, C D, D E, E A, are equal to each other. Whereſore A B C D E is an equilateral Pentagon. I ſay, it is alſo equiangular: For becauſe the Circumference A B is equal to the Circumference D E; by adding the Circumference B C D, which is common, the whole Circumference A B C D is equal to the whole Circumference E D C B: But the Angle A E D ſtands on the Circumference A B C D, and B A E on the Circumference E D C B; therefore the Angle B A E is equal to the Angle A E D. For the ſame Reaſon, each of the other Angles A B C, B C D, C D E, is equal to B A E, or A E D; wherefore the Pentagon A B C D E is equiangular. But it has been proved to be alſo equilateral: And, conſequently, *there is an equilateral and equiangular Pentagon inſcribed in a given Circle*; which was to be done.

PROPOSITION XII.

Problem.

To deſcribe an equilateral and equiangular Pentagon about a Circle given.

LET A B C D E be the given Circle. It is required to deſcribe an equilateral and equiangular Pentagon about the ſame.

Let A, B, C, D, E, be the angular Points of a Pentagon ſuppoſed to be inſcribed * in the Circle; ſo that the Circumferences A B, B C, C D, D E, E A, be * By 11 of this.

equal; and let the Right Lines GH, HK, KL, LM,
† 17. 3. MG, be drawn, touching † the Circle in the Points
A, B, C, D, E: Let F be the Centre of the Circle
ABCDE; and join FB, FK, FC, FL, FD.

Then, becauſe the Right Line KL touches the
Circle ABCDE in the Point C, and the Right Line
FC is drawn from the Centre F to C, the Point of
‡ 18. 3. Contact; FC will be ‡ perpendicular to KL; and
ſo both the Angles at C are Right Angles. For the
ſame Reaſon, the Angles at the Points B, D, are
Right Angles. And becauſe FCK is a Right Angle,
* 47. 1. the Square of FK will be * equal to the Squares
of FC, CK: And for the ſame Reaſon, the Square
of FK is equal to the Squares of FB, BK. Therefore
the Squares of FC, CK, are equal to the Squares
of FB, BK. But the Square of FC is equal to the
Square of FB: Wherefore the Square of CK ſhall be
equal to the Square of BK; and ſo BK is equal to
CK. And becauſe FB is equal to FC, and FK is
common, the two Sides BF, FK, are equal to the
two Sides CF, FK, and the Baſe BK is equal to the
† 8. 1. Baſe KC; and ſo the Angle BFK ſhall be † equal
to the Angle KFC, and the Angle BKF to the Angle
FKC. Therefore the Angle BFC is double to
the Angle KFC, and the Angle BKC double to the
Angle FKC: For the ſame Reaſon the Angle CFD
is double to the Angle CFL, and the Angle CLD
double to the Angle CLF. And becauſe the Circumference
BC is equal to the Circumference CD,
‡ 27. 3. the Angle BFC ſhall be ‡ equal to the Angle CFD.
But the Angle BFC is double to the Angle KFC,
and the Angle DFC double to LFC. Therefore
the Angle KFC is equal to the Angle CFL. And
ſo FKC, FLC, are two Triangles, having two Angles
of the one equal to two Angles of the other, each
to each, and one Side of the one equal to one Side of
the other, *viz.* the common Side FC; wherefore they
† 26. 1. ſhall have † the other Sides of the one equal to the
other Sides of the other; and the other Angle of the
one equal to the other Angle of the other. Therefore
the Right Line KC is equal to the Right Line
CL, and the Angle FKC to the Angle FLC. And
ſince KC is equal to CL, KL ſhall be double to KC.
And by the ſame Reaſon, we prove that HK is double
to

to B K. Again, becauſe B K has been proved equal to K C, and K L the double of K C, as alſo H K the double of B K; H K ſhall be equal to K L. So likewiſe we prove, that G H, G M and M L, are each equal to H K, or K L: Therefore the Pentagon G H K L M is equilateral. I ſay, alſo it is equiangular. For becauſe the Angle F K C is equal to the Angle F L C, and the Angle H K L has been proved to be double to the Angle F K C; and alſo K L M double to F L C: Therefore the Angle H K L ſhall be equal to the Angle K L M. By the ſame Reaſon we demonſtrate, that every one of the Angles K H G, H G M, G M L, is equal to the Angle H K L, or K L M. Therefore the five Angles, G H K, H K L, K L M, L M G, M G H, are equal between themſelves. And ſo, *the Pentagon* G H K L M *is equiangular; and it has been proved likewiſe to be equilateral, and deſcribed about the Circle* A B C D E; which was to be done.

PROPOSITION XIII.

PROBLEM.

To deſcribe a Circle in an equilateral and equiangular Pentagon.

LET A B C D E be an equilateral and equiangular Pentagon. It is required to inſcribe a Circle in the ſame.

Biſect * the Angles B C D, C D E, by the Right Lines C F, D F; and from the Point F, wherein C F, D F, meet each other, let the Right Lines F B, F A, F E, be drawn. Now, becauſe B C is equal to C D, and C F is common, the two Sides B C, C F, are equal to the two Sides D C, C F; and the Angle B C F is equal to the Angle D C F. Therefore the Baſe B F is † equal to the Baſe F D; and the Triangle B F C equal to the Triangle D C F, and the other Angles of the one equal to the other Angles of the other, which are ſubtended by the equal Sides: Therefore the Angle C B F ſhall be equal to the Angle C D F. And becauſe the Angle C D E is double to the Angle C D F, and the Angle C D E is equal to the Angle A B C, as alſo C D F equal to C B F;

* 9. 1.

† 4. 1.

the Angle C B A will be double to the Angle C B F; and ſo the Angle A B F equal to the Angle C B F: Wherefore the Angle A B C is biſected by the Right Line B F. After the ſame manner we prove, that either of the Angles B A E, or A E D, is biſected by the Right Line A F, or F E. From the Point F draw
* 12. 1. * F G, F H, F K, F L, F M, perpendicular to the Right Lines A B, B C, C D, D E, E A: Then, ſince the Angle H C F is equal to the Angle K C F, and the Right Angle F H C equal to the Right Angle F K C; the two Triangles F H C, F K C, ſhall have two Angles of the one equal to two Angles of the other, and one Side of the one equal to one Side of the other, *viz.* the Side F C common to each of them:
† 26. 1. And ſo the other Sides of the one will be † equal to the other Sides of the other, and the Perpendicular F H equal to the Perpendicular F K. In the ſame manner we demonſtrate, that F L, F M, or F G, is equal to F H, or F K. Therefore the five Right Lines F G, F H, F K, F L, F M, are equal to each other, and ſo a Circle deſcribed on the Centre F, with either of the Diſtances F G, F H, F K, F L, F M, will paſs thro' the other Points, and ſhall touch the Right Lines A B, B C, C D, D E, E A; ſince the Angles at G, H, K, L, M, are Right Angles. For, if it does not touch them but cuts them, a Right Line drawn from the Extremity of the Diameter of a Circle, at Right Angles to the Diameter, will fall within
‡ 16. 3. the Circle; which is ‡ abſurd. Therefore, *a Circle deſcribed on the Centre* F, *with the Diſtance of any one of the Points* G, H, K, L, M, *will not cut the Right Lines* A B, B C, C D, D E, E A, *and ſo will neceſſarily touch them*; which was to be done.

Coroll. If two of the neareſt Angles of an equilateral and equiangular Figure be biſected, and, from the Point in which the Lines biſecting the Angles meet, there be drawn Right Lines to the other Angles of the Figure, all the Angles of the Figure will be biſected.

PROPOSITION XIV.

Problem.

To deſcribe a Circle about a given equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to deſcribe a Circle about the ſame.

Biſect both the Angles BCD, CDE, by the Right Lines CF, FD; and draw FB, FA, FE, from the Point F, in which they meet. Then each of the other Angles CBA, BAE, AED, ſhall be biſected * by the Right Lines BF, FA, FE. And ſince the Angle BCD is equal to the Angle CDE, and the Angle FCD is half the Angle BCD; as likewiſe CDF, half CDE; the Angle FCD will be equal to the Angle FDC; and ſo the Side CF † equal to the Side FD. We demonſtrate, in like manner, that FB, FA, or FE, is equal to FC, or FD. Therefore the five Right Lines FA, FB, FC, FD, FE, are equal to each other. And ſo, *a Circle being deſcribed on the Centre* F, *with any of the Diſtances* FA, FB, FC, FD, FE, *will paſs thro' the other Points, and will be deſcribed about the equilateral and equiangular Pentagon* ABCDE; which was to be done.

* Cor. of preced.

† 6. 1.

PROPOSITION XV.

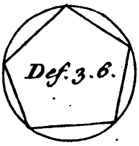
Problem.

To inſcribe an equilateral and equiangular Hexagon in a given Circle.

LET ABCDEF be a Circle given. It is required to inſcribe an equilateral and equiangular Hexagon therein.

Draw AD, a Diameter of the Circle ABCDEF, and let G be the Centre; and about the Point D, as a Centre, with the Diſtance DG, let a Circle, EGCH, be deſcribed; join EG, GC, which produce to the Points B, F: Likewiſe join AB, BC, CD, DE, EF, F A: I ſay, ABCDEF is an equilateral and equiangular Hexagon.

For, ſince the Point G is the Centre of the Circle
A B C D E F, G E will be equal to G D. Again, be-
cauſe the Point D is the Centre of the Circle E G C H,
D E ſhall be equal to D G: But G E has been proved
equal to G D; therefore G E is equal to E D. And ſo
E G D is an equilateral Triangle; and conſequently
the three Angles thereof, E G D, G D E, D E G, are
* Cor. 5. 1. * equal between themſelves. But the three Angles of
† 32. 1. a Triangle are † equal to two Right Angles; there-
fore the Angle E G D is a third Part of two Right
Angles. In the ſame manner we demonſtrate, that
D G C is one third Part of two Right Angles: And
ſince the Right Line C G, ſtanding upon the Right
‡ 13. 1. Line E B, makes ‡ the adjacent Angles E G C, C G B;
therefore the other Angle, C G B, is alſo one third
Part of two Right Angles. Therefore the Angles
E G D, D G C, C G B, are equal between them-
ſelves: And the Angles that are vertical to them, *viz.*
* 15. 1. the Angles B G A, A G F, F G E, are * equal to the
Angles E G D, D G C, C G B. Wherefore the ſix
Angles EGD, DGC, CGB, BGA, AGF, FGE,
† 26. 3. are equal to one another. But equal Angles ſtand †
on equal Circumferences: Therefore the ſix Circum-
ferences A B, B C, C D, D E, E F, F A, are equal to
‡ 29. 3. each other. But equal Right Lines ſubtend ‡ equal
Circumferences: Therefore the ſix Right Lines are
equal between themſelves; and accordingly the Hex-
agon A B C D E F is equilateral. I ſay, it is alſo
equiangular. For, becauſe the Circumference A F is
equal to the Circumference E D, add the common
Circumference A B C D, and the whole Circumference
F A B C D is equal to the whole Circumference
E D C B A. But the Angle F E D ſtands on the
Circumference F A B C D; and the Angle A F E,
on the Circumference E D C B A: Therefore the An-
* 27. 3. gle A F E is * equal to the Angle D E F. In the
ſame manner we prove, that the other Angles of the
Hexagon A B C D E F are ſeverally equal to A F E,
or F E D. Therefore, *the Hexagon* A B C D E F *is
equiangular. But it has been proved to be alſo equilateral,
and is inſcribed in the Circle* A B C D E F; which was
to be done.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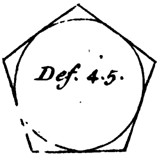
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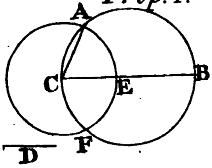

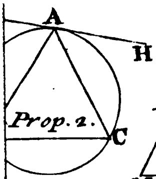
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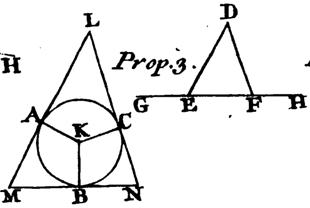
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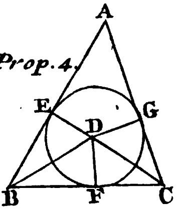

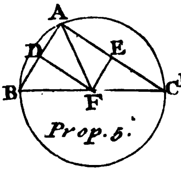
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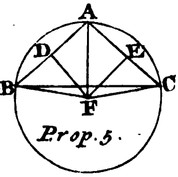
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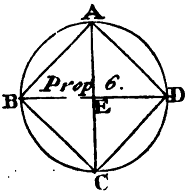
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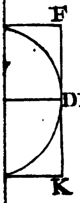

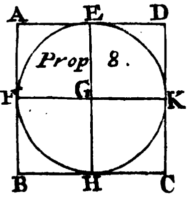
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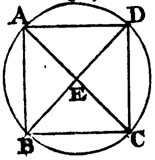

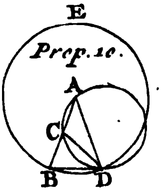
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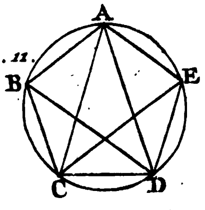

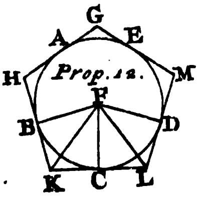
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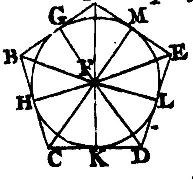

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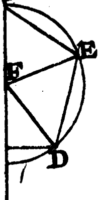

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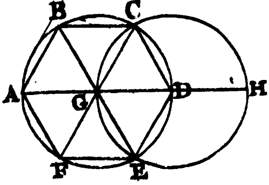

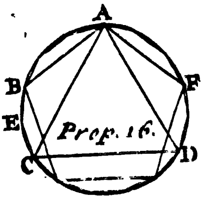
Prop. 16.

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Coroll. From hence it is manifeſt, that the Side of the Hexagon is equal to the Semidiameter of the Circle. And if we draw, thro' the Points A, B, C, D, E, F, Tangents to the Circle, an equilateral and equiangular Hexagon, will be deſcribed about the Circle, as is manifeſt, from what has been ſaid concerning the Pentagon. And ſo likewiſe may a Circle be inſcribed and circumſcribed about a given Hexagon; *which was to be done.*

PROPOSITION XVI.

PROBLEM.

To deſcribe an equilateral and equiangular Quindecagon in a given Circle.

LET A B C D be a Circle given. It is required to deſcribe an equilateral and equiangular Quindecagon in the ſame.

Let A C be the Side of an equilateral Triangle inſcribed in the Circle A B C D, and A B the Side of a Pentagon. Now, if the whole Circumference of the Circle A B C D be divided into fifteen equal Parts, the Circumference A B C, one Third of the Whole, ſhall be five of the ſaid fifteen equal Parts; and the Circumference A B, one Fifth of the Whole, will be three of the ſaid Parts: Wherefore the remaining Circumference B C will be two of the ſaid Parts. And if B C be biſected in the Point E, then B E, or E C, will be one fifteenth Part of the whole Circumference A B C D. And ſo, *if* B E, E C, *be joined, and either* E C, *or* E B, *be continually applied in the Circle, there ſhall be an equilateral and equiangular Quindecagon deſcribed in the Circle* A B C D; which was to be done.

If, according to what hath been ſaid of the Pentagon, Right Lines are drawn thro' the Diviſions of the Circle touching the ſame, there will be deſcribed about the Circle an equilateral and equiangular Quindecagon. And, moreover, a Circle may be inſcribed, or circumſcribed, about a given equilateral and equiangular Quindecagon.

EUCLID's

ELEMENTS,

BOOK V.

DEFINITIONS.

I. *A Part is a Magnitude of a Magnitude, the Leſs of the Greater, when the Leſſer meaſures the Greater.*

II. *But a Multiple is a Magnitude of a Magnitude, the Greater of the Leſſer, when the Leſſer meaſures the Greater.*

III. *Ratio is a certain mutual Habitude of Magnitudes of the ſame Kind, according to Quantity.*

IV. *Magnitudes are ſaid to have Proportion to each other, which, being multiplied, can exceed one another.*

V. *Magnitudes are ſaid to be in the ſame Ratio, the firſt to the ſecond, and the third to the fourth; when the Equimultiples of the firſt and third, compared with the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, are either both together, greater, equal, or leſs, than the Equimultiples of the ſecond and fourth, if thoſe be taken that anſwer each other.*

That is, if there be four Magnitudes, and you take any Equimultiples of the firſt and third, and alſo any Equimultiples of the ſecond and fourth; and if the Multiple of the firſt be greater than the Multiple of the ſecond; and alſo the Multiple of the third greater than the Multiple of the fourth; or, if the Multiple of the firſt be equal to the Multiple of the ſecond; and alſo the Multiple of the third equal to the Multiple of the fourth; or, laſtly, if the Multiple of the firſt be leſs than the Multiple of the ſecond; and alſo that of the third leſs than that of the fourth, and theſe Things happen according to every Multiplication whatſoever: Then the four Magnitudes are in the ſame Ratio; the firſt to the ſecond, as the third to the fourth.

VI. *Magnitudes that have the ſame Proportion, are called Proportionals.*

Expounders uſually lay down here that Definition, for Magnitudes, which *Euclid* has given for Numbers, only, in his Seventh Book, *viz.* That

Numbers are proportional, when the firſt is either the ſame Multiple of the ſecond, as the third is of the fourth, or elſe the ſame Part, or Parts.

But this Definition appertains only to Numbers, and commenſurable Quantities; and ſo, ſince it is not univerſal, *Euclid* did well to reject it in this Element, which treats of the Properties of all Proportionals; and to ſubſtitute another general one, agreeing to all Kinds of Magnitudes. In the mean Time, Expounders very much endeavour to demonſtrate the Definition here laid down by *Euclid*, by the uſual received Definition of proportional Numbers; but this much eaſier flows from that, than that from this; which may be thus demonſtrated:

Firſt, Let A, B, C, D, be four Magnitudes, which are in the ſame Ratio, according to the Conditions that Magnitudes in the ſame Ratio muſt have according to the fifth Definition; and let the firſt be a Multiple of the ſecond: I ſay, the third is alſo the ſame Multiple of the fourth. For Example: Let A be equal to 5B: Then C ſhall be equal to 5D. Take any

Number, for Example, 2, by which let 5 be multiplied, and the Product will be 10: And let 2A, 2C, be Equimultiples of the firſt and third Magnitudes A and C: Alſo, let 10B and 10D be Equimultiples of the ſecond and fourth Magnitudes B and D. Then (by *Def.* 5.) if 2A be equal to 10B, 2C ſhall be equal to 10D. But ſince A (from the *Hypotheſis*) is five Times B, 2A ſhall be equal to 10B; and ſo 2C equal to 10D, and C equal to 5D; that is, C will be five Times D. W. W. D.

A : B : : C : D

2A, 10B, 2C, 10D

Secondly, Let A be any Part of B; then C will be the ſame Part of D. For, becauſe A is to B, as C is to D; and ſince A is ſome Part of B; then B will be a Multiple of A: And ſo (by *Caſe* 1.) D will be the ſame Multiple of C; and accordingly C ſhall be the ſame Part of the Magnitude D, as A is of B. W. W. D.

Thirdly, Let A be equal to any Number of whatſoever Parts of B. I ſay, C is equal to the ſame Number of the like Parts of D. For Example: Let A be a fourth Part of five Times B; that is, let A be equal to $\frac{5}{4}$B. I ſay, C is alſo equal to $\frac{5}{4}$D. For, becauſe A is equal $\frac{5}{4}$B, each of them being multiplied by 4, then 4A will be equal to 5B. And ſo, if the Equimultiples of the firſt and third, *viz.* 4A, 4C, be aſſumed; as alſo the Equimultiples of the ſecond and fourth, *viz.* 5B, 5D; and (by the *Definition*) if 4A is equal to 5B; then 4C is equal to 5D. But 4A has been proved equal to 5B, and ſo 4C ſhall be equal to 5D, and C equal to $\frac{5}{4}$D. W. W. D.

A : B : : C : D

4A, 5B, 4C, 5D

And univerſally, if A be equal to $\frac{n}{m}$B, C will be equal to $\frac{n}{m}$D. For let A and C be multiplied by *m*, and B and D by *n*. And becauſe A is equal to $\frac{n}{m}$B; *m*A ſhall be equal to *n*B; wherefore (by *Def.* 5) *m*C will be equal to *n*D, and C equal to $\frac{n}{m}$D. W. W. D.

A : B : : C : D

*m*A, *n*B, *m*C, *n*D

VII. *When, of Equimultiples, the Multiple of the firſt exceeds the Multiple of the ſecond, but the Multiple of the third does not exceed the Multiple of the fourth; then the firſt to the ſecond is ſaid to have a greater Proportion, than the third to the fourth.*

VIII. *Analogy is a Similitude of Proportions.*

IX. *Analogy at leaſt conſiſts of three Terms.*

X. *When three Magnitudes are Proportionals, the firſt is ſaid to have, to the third, a duplicate Ratio to what it has to the ſecond.*

XI. *But when four Magnitudes are continued Proportionals, the firſt ſhall have a triplicate Ratio to the fourth of what it has to the ſecond; and ſo always one more in Order, as the Proportionals ſhall be extended.*

XII. *Homologous Magnitudes, or Magnitudes of a like Ratio, are ſaid to be ſuch whoſe Antecedents are to the Antecedents, and Conſequents to the Conſequents.*

XIII. *Alternate Ratio is the comparing of the Antecedent with the Antecedent, and the Conſequent with the Conſequent.*

XIV. *Inverſe Ratio is, when the Conſequent is taken as the Antecedent, and ſo compared with the Antecedent as a Conſequent.*

XV. *Compounded Ratio is, when the Antecedent and Conſequent, taken both as one, is compared to the Conſequent itſelf.*

XVI. *Divided Ratio is, when the Exceſs, whereby the Antecedent exceeds the Conſequent, is compared with the Conſequent.*

XVII. *Converſe Ratio is, when the Antecedent is compared with the Exceſs, by which the Antecedent exceeds the Conſequent.*

XVIII. *Ratio of Equality is, where there are taken more than two Magnitudes in one Order, and a* *like Number of Magnitudes in another Order, comparing two to two being in the ſame Proportion; and it ſhall be in the firſt Order of Magnitudes, as the firſt is to the laſt, ſo in the ſecond Order of Magnitudes is the firſt to the laſt: Or otherwiſe, it is the Compariſon of the Extremes together, the Means being omitted.*

XIX. *Ordinate Proportion is, when as the Antecedent is to the Conſequent, ſo is the Antecedent to the Conſequent; and as the Conſequent is to any other, ſo is the Conſequent to any other.*

XX. *Perturbate Proportion is, when there are three or more Magnitudes, and others alſo, that are equal to theſe in Multitude, as in the firſt Magnitudes the Antecedent is to the Conſequent; ſo in the ſecond Magnitudes is the Antecedent to the Conſequent: And as in the firſt Magnitudes the Conſequent is to ſome other, ſo in the ſecond Magnitudes is ſome other, to the Antecedent.*

AXIOMS.

I. *Equimultiples of the ſame, or of equal Magnitudes, are equal to each other.*

II. *Thoſe Magnitudes that have the ſame Equimultiple, or whoſe Equimultiples are equal, are equal to each other.*

PRO-

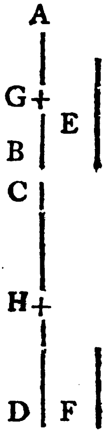
PROPOSITION I.

THEOREM.

If there be any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each of each; whatſoever Multiple any one of the former Magnitudes is of its correſpondent one, the ſame Multiple are all the former Magnitudes of all the latter.

LET there be any Number of Magnitudes A B, C D Equimultiples of a like Number of Magnitudes E, F, each of each. I ſay, what Multiple the Magnitude A B is of E, the ſame Multiple A B, and C D, together, is of E and F together.

For, becauſe A B and C D are Equimultiples of E and F, as many Magnitudes equal to E, that are in A B, ſo many ſhall be equal to F in C D. Now, divide A B into Parts equal to E, which let be A G, G B; and C D into Parts equal to F, *viz.* C H, H D. Then the Multitude of Parts, C H, H D, ſhall be equal to the Multitude of Parts A G, G B. And ſince A G is equal to E, and C H to F; A G and C H, together, ſhall be equal to E and F together. By the ſame Reaſon, becauſe G B is equal to E, and H D to F, G B and H D, together, will be equal to E and F together. Therefore, as often as E is contained in A B, ſo often is E and F, together, contained in A B and C D, together. And ſo as often as F is contained in C D, ſo often are E and F, together, contained in A B, and C D together. Therefore, *if there are any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each of each; whatſoever Multiple any one of the former Magnitudes is of its correſpondent one, the ſame Multiple are all the former Magnitudes of all the latter*; which was to be demonſtrated.

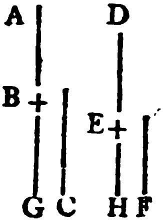


PROPOSITION II.

THEOREM.

If the firſt be the ſame Multiple of the ſecond, as the third is of the fourth; and if the fifth be the ſame Multiple of the ſecond, as the ſixth is of the fourth; then ſhall the firſt, added to the fifth, be the ſame Multiple of the ſecond, as the third, added to the ſixth, is of the fourth.

LET the firſt A B be the ſame Multiple of the ſecond C, as the third D E is of the fourth F; and let the fifth B G be the ſame Multiple of the ſecond C, as the ſixth E H is of the fourth F. I ſay, the firſt added to the fifth, *viz.* A G, is the ſame Multiple of the ſecond C, as the third added to the ſixth, *viz.* D H, is of the fourth F.



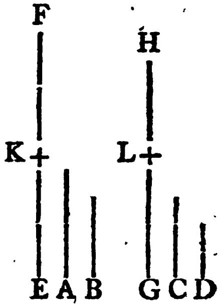
For, becauſe A B is the ſame Multiple of C, as D E is of F; there are as many Magnitudes equal to C in A B, as there are Magnitudes equal to F in D E. And, for the ſame Reaſon, there are as many Magnitudes equal to C in B G, as there are Magnitudes equal to F in E H. Therefore there are as many Magnitudes equal to C, in the whole A G, as there are Magnitudes equal to F in D H. Wherefore A G is the ſame Multiple of C, as D H is of F. And ſo the firſt, added to the fifth, A G, is the ſame Multiple of the ſecond C, as the third, added to the ſixth, D H, is of the fourth F. Therefore, *if the firſt be the ſame Multiple of the ſecond, as the third is of the fourth; and if the fifth be the ſame Multiple of the ſecond, as the ſixth is of the fourth; then ſhall the firſt, added to the fifth, be the ſame Multiple of the ſecond, as the third, added to the ſixth, is of the fourth*; which was to be demonſtrated.

PROPOSITION III.

THEOREM.

If the firſt be the ſame Multiple of the ſecond, as the third is of the fourth, and there be taken Equimultiples of the firſt and third; then will the Magnitudes ſo taken be Equimultiples of the ſecond and fourth.

LET the firſt A be the ſame Multiple of the ſecond B, as the third C is of the fourth D; and let E F, G H, be Equimultiples of A and C. I ſay, E F, is the ſame Multiple of B as G H is of D.



For, becauſe E F is the ſame Multiple of A, as G H is of C, there are as many Magnitudes equal to A in E F, as there are Magnitudes equal to C in G H. Now divide E F into the Magnitudes E K, K F, each equal to A, and G H into the Magnitudes G L, L H, each equal to C. Then the Number of the Magnitudes E K, K F, will be equal to the Number of the Magnitudes G L, L H. And becauſe A is the ſame Multiple of B, as C is of D, and E K is equal to A, and G L to C; E K will be the ſame Multiple of B, as G L is of D. For the ſame Reaſon, K F ſhall be the ſame Multiple of B, as L H is of D. Therefore becauſe the firſt E K is the ſame Multiple of the ſecond B, as the third G L is of the fourth D, and K F the fifth, is the ſame Multiple of B, the ſecond, that L H, the ſixth, is of D the fourth: Therefore the firſt added to the fifth, E F, ſhall be * the ſame Multiple of the ſecond B, as the third added to the ſixth, G H, is of the fourth D. *If*, therefore, *the firſt be the ſame Multiple of the ſecond, as the third is of the fourth, and there be taken Equimultiples of the firſt and third; then will the Magnitudes ſo taken, be Equimultiples of the ſecond and fourth*; which was to be demonſtrated.

* 2 *of this.*

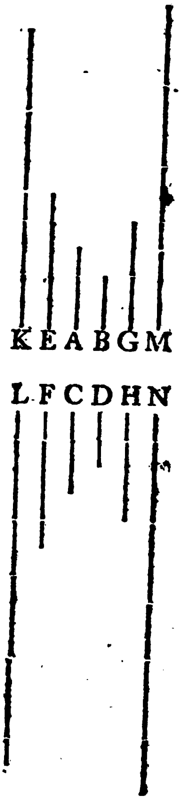
PRO-

PROPOSITION IV.

Theorem.

If the firſt have the ſame Proportion to the ſecond, as the third to the fourth; then alſo ſhall the Equimultiples of the firſt and third have the ſame Proportion to the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, if they be ſo taken as to anſwer each other.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C hath to the fourth D; and let E and F, the Equimultiples of A and C, be any how taken; as alſo G and H, the Equimultiples of B and D. I ſay, E is to G as F is to H.



For take K and L, any Equimultiples of E and F; and alſo M and N, any, of G and H.

Then, becauſe E is the ſame Multiple of A, as F is of C, and K and L are taken Equimultiples of E and F; therefore K will be * the ſame Multiple of A, as L is of C. For the ſame Reaſon, M is the ſame Multiple of B, as N is of D. And ſince A is to B, as C is to D, and K and L are Equimultiples of A and C; and alſo M and N Equimultiples of B and D; if K exceeds M, then † L will exceed N; if K be equal to M, L will be equal to N; and if K is leſs than M, then L will be leſs than N: But K and L are Equimultiples of E and F, alſo M and N are Equimultiples of G and H. Therefore, as E is to G, ſo, ſhall ‡ F be to H. Wherefore, *if the firſt have the ſame Proportion to the ſecond, as the third to the fourth; then alſo ſhall the Equi-*

* 3 *of this.*

† *Def.* 5. *of this.*

‡ *Def.* 5.

multiples of the firſt and third have the ſame Proportion to the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, if they be ſo taken as to anſwer each other; which was to be demonſtrated.

Becauſe it is demonſtrated, if K exceeds M, then L will exceed N; and if K be equal to M, L will be equal to N; and if K be leſs than M, L will be leſs than N: It is manifeſt, likewiſe, if M exceeds K, that N ſhall exceed L; if equal, equal; but if leſs, leſs. And therefore, as G is to E, ſo is * H to F. * *Def.* 5.

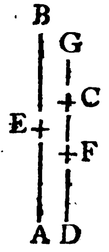
Coroll. From hence it is manifeſt, if four Magnitudes be proportional, that they will be alſo inverſely proportional.

PROPOSITION V.

THEOREM.

If one Magnitude be the ſame Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Reſidue of the one ſhall be the ſame Multiple of the Reſidue of the other, as the Whole is of the Whole.

LET the Magnitude AB be the ſame Multiple of the Magnitude CD, as the Part taken away AE, is of the Part taken away CF. I ſay, that the Reſidue EB is the ſame Multiple of the Reſidue FD, as the Whole AB is of the Whole CD.



For, let EB be ſuch a Multiple of CG, as AE is of CF.

Then, becauſe AE is the ſame Multiple of CF, as EB is of CG, AE will be * the ſame Multiple of CF, as AB is of GF. But AE and AB are put Equimultiples of CF and CD: Therefore AB is the ſame Multiple of GF, as of CD; and ſo GF is † equal to CD. Now, let CF, which is common, be taken away; then the Reſidue GC is equal

* 1 *of this.*

† *Ax.* 2 *of this.*

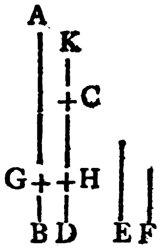
equal to the Reſidue D F. And then, becauſe A E is the ſame Multiple of C F, as E B is of C G, and C G is equal to D F; A E ſhall be the ſame Multiple of C F, as E B is of F D. But A E is put the ſame Multiple of C F, as A B is of C D: Therefore E B is the ſame Multiple of F D, as A B is of C D; and ſo the Reſidue E B is the ſame Multiple of the Reſidue F D, as the Whole A B is of the Whole C D. Wherefore, *if one Magnitude be the ſame Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Reſidue of the one ſhall be the ſame Multiple of the Reſidue of the other, as the Whole is of the Whole*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

If two Magnitudes be Equimultiples of two Magnitudes, and ſome Magnitudes Equimultiples of the ſame, be taken away; then the Reſidues are either equal to thoſe Magnitudes, or elſe Equimultiples of them.

LET two Magnitudes A B, C D, be Equimultiples of two Magnitudes E, F; and let the Magnitudes A G, C H, Equimultiples of the ſame, E, F, be taken from A B, C D: I ſay, the Reſidues G B, H D, are either equal to E, F, or are Equimultiples of them.

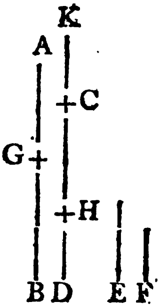


For, firſt, Let G B be equal to E. I ſay, H D is elſo equal to F. For let C K be equal to F. Then, becauſe A G is the ſame Multiple of E, as C H is of F; and G B is equal to E; and C K to F; A B will be * the ſame Multiple of E, as K H is of F. But A B and C D are put Equimultiples of E and F. Therefore K H is the ſame Multiple of F, as C D is of F.

* 1 *of this.*

And becauſe K H and C D are Equimultiples of F; K H will be equal to C D. Take away C H, which is common; then the Reſidue K C is equal to the Reſidue H D. But K C is equal to F. Therefore H D is equal to F; and ſo G B ſhall be equal to E, and H D to F.

In like manner we demonſtrate, if G B was a Multiple of E, that H D is a like Multiple of F. Therefore, *if two Magnitudes be Equimultiples of two Magnitudes, and ſome Magnitudes, Equimultiples of the ſame, be taken away; then the Reſidues are either equal to thoſe Magnitudes, or elſe Equimultiples of them;* which was to be demonſtrated.



PROPOSITION VII.

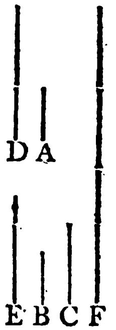
Theorem.

Equal Magnitudes have the ſame Proportion to the ſame Magnitude; and one and the ſame Magnitude have the ſame Proportion to equal Magnitudes.

LET A, B, be equal Magnitudes, and let C be any other Magnitude. I ſay, A and B have the ſame Proportion to C; and likewiſe C has the ſame Proportion to A as to B.

For take D, and E, Equimultiples of A and B; and let F be any other Multiple of C.

Now, becauſe D is the ſame Multiple of A, as E is of B, and A is equal to B, D ſhall be alſo equal to E; but F is a Magnitude taken at Pleaſure. Therefore if D exceeds F, then E will exceed F; if D be equal to F, E will be equal to F; and if leſs, leſs. But D and E are Equimultiples of A and B; and F is any other Multiple of C. Therefore it will be * as A is to C, ſo is B to C. * *Def.* 5.



I ſay, moreover, that C has the ſame Proportion to A as to B. For the ſame Conſtruction remaining, we prove, in like manner, that D is equal to E. Therefore, if F exceeds D, it will alſo exceed E; if it be equal to D, it will be equal to E; and if it be leſs than D, it will be leſs than E. But F is a Multiple of C; and D and E, any other Equimultiples of A and B; therefore, as C is to A, ſo ſhall * C be to B. Wherefore, *equal Magnitudes have the ſame Proportion to the ſame Magnitude, and the ſame Magnitude to equal ones*; which was to be demonſtrated.

* Def. 5.

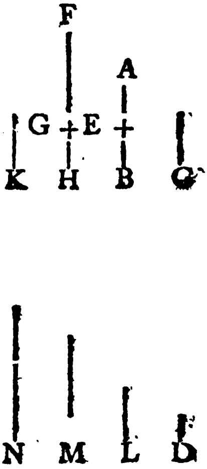
PROPOSITION VIII.

Theorem.

The greater of any two unequal Magnitudes has a greater Proportion to ſome third Magnitude, than the leſs has; and that third Magnitude hath a greater Proportion to the leſſer of the two Magnitudes, than it has to the greater.

LET AB and C be two unequal Magnitudes, whereof AB is the greater; and let D be any third Magnitude. I ſay, AB has a greater Proportion to D, than C has to D; and D has a greater Proportion to C, than it has to AB.

Becauſe AB is greater than C, make BE equal to C, that is, let AB exceed C by AE; then AE, multiplied ſome Number of Times, will be greater than D. Now let AE be multiplied until it exceeds D, and let that Multiple of AE greater than D be FG. Make GH the ſame Multiple of EB, and K of C, as FG is of AE. Alſo, aſſume L double to D, M triple, and ſo on, until ſuch a Multiple of D is had, as is the neareſt greater than K; let this be N, and let M be a Multiple of D, the neareſt leſs than N.



Now, becauſe N is the neareſt Multiple of D greater

than K; M will not be greater than K; that is, K
will not be leſs than M. And ſince F G is the ſame
Multiple of A E, as G H is of E B; F G ſhall be * * 1 of this.
the ſame Multiple of A E, as F H is of A B; but F G
is the ſame Multiple of A E, as K is of C: Where-
fore F H is the ſame Multiple of A B as K is of C;
that is, F H and K are Equimultiples of A B and C.
Again, becauſe G H is the ſame Multiple of E B, as K
is of C, and E B is equal to C; G H ſhall be † equal † Ax. 1.
to K. But K is not leſs than M: Therefore G H
ſhall not be leſs than M. But F G is greater than D:
Therefore the whole F H will be greater than M and
D; but M and D, together, are equal to N, becauſe
M is a Multiple of D, the neareſt leſſer than N:
Wherefore F H is greater than N. And ſo, ſince F H
exceeds N, and K does not; and F H and K are
Equimultiples of A B and C, and N is another Mul-
tiple of D; therefore A B will have ‡ a greater Ratio ‡ Def. 7.
to D, than C has to D. I ſay, moreover, that D has
a greater Ratio to C than it has to A B: For the
ſame Conſtruction remaining, we demonſtrate, as be-
fore, that N exceeds K, but not F H. And N is a
Multiple of D, and F H and K are Equimultiples of
A B and C. Therefore D has a greater Proportion
to C, than D hath to A B. Wherefore, *the greater of
any two unequal Magnitudes has a greater Proportion to
ſome third Magnitude than the leſs has; and the third
Magnitude hath a greater Proportion to the leſſer of the
two Magnitudes than it has to the greater*; which was
to be demonſtrated.

PROPOSITION IX.

THEOREM.

Magnitudes which have the ſame Proportion to one and the ſame Magnitude, are equal to one another; and if a Magnitude has the ſame Proportion to other Magnitudes, theſe Magnitudes are equal to one another.

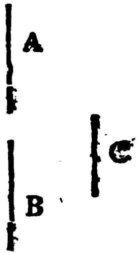
LET the Magnitudes A and B have the ſame Proportion to C. I ſay, A is equal to B.

For,

* 8 of this. For, if it was not, A and B would not * have the ſame Proportion to the ſame Magnitude C; but they have. Therefore A is equal to B.

Again, let C have the ſame Proportion to A as to B. I ſay, A is equal to B.

For, if it be not, C will not * have the ſame Proportion to A as to B; but it hath: Therefore A is neceſſarily equal to B. Therefore, *Magnitudes that have the ſame Proportion to one and the ſame Magnitude, are equal to one another; and, if a Magnitude, has the ſame Proportion to other Magnitudes, theſe Magnitudes are equal to one another*; which was to be demonſtrated.



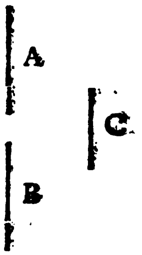
PROPOSITION X.

THEOREM.

Of Magnitudes having Proportion to the ſame Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the ſame bears a greater Proportion, is the leſſer Magnitude.

LET A have a greater Proportion to C, than B has to C. I ſay, A is greater than B.

For, if it be not greater, it will either be equal or leſs. But A is not equal to B, becauſe
* 7 of this. then both A and B would have * the ſame Proportion to the Magnitude C; but they have not. Therefore A is not equal to B: Neither is it leſs than
† 8 of this. B; for then A would have † a leſs Proportion to C, that B would have; but it hath not a leſs Proportion. Therefore A is not leſs than B. But it has been proved likewiſe not to be equal to it: Therefore A ſhall be greater than B.



Again, let C have a greater Proportion to B than to A. I ſay, B is leſs than A.

For, if it be not leſs, it is greater, or equal. Now, B is not equal to A, for then C would have * the ſame Proportion to A as to B; but this it has not. *7 *of this.* Therefore A is not equal to B; neither is B greater than A; for if it was, C would have † a leſs Proportion to B than to A; but it has not: Therefore B is not greater than A. † 8 *of this.* But it has alſo been proved not to be equal to it. Wherefore B ſhall be leſs than A. Therefore, *of Magnitudes having Proportion to the ſame Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the ſame bears a greater Proportion, is the leſſer Magnitude*; which was to be demonſtrated.

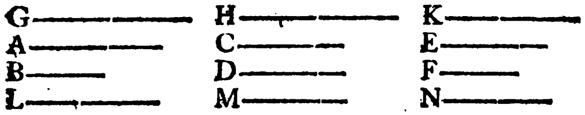
PROPOSITION XI.

Problem.

Proportions that are one and the ſame to any third, are alſo the ſame to one another.

LET A be to B, as C is to D; and C to D, as E to F. I ſay, A is to B, as E is to F.

For, take G, H, and K, Equimultiples of A, C, and E; and L, M, and N, other Equimultiples of B, D, and F. Then, becauſe A is to B, as C is to D, and there are taken G and H, the Equimultiples of A and C, and L and M, any other Equimultiples of B and D; if G exceeds L, * then H will exceed M; and if G be equal to L, H will be equal to M; and if leſs, leſs. * Def. 5. of this. Again, becauſe as C is to D, ſo is E to F; and H and K are taken Equimultiples of C and E; as likewiſe M and N, any other Equimultiples of D and F; if H exceeds M *, then K will exceed N; and if H be equal to M, K will be equal to N; and if leſs, leſs. But if H exceeds M, G will alſo exceed L; if equal, equal; and if leſs, leſs. Wherefore, if G exceeds L, K will alſo exceed N; and if G be equal to



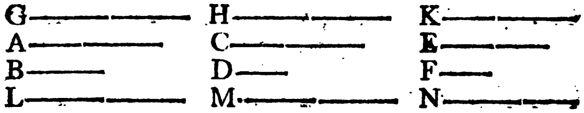
L, K will be equal to N; and if leſs, leſs. But G and K are Equimultiples of A and E; and L and N are Equimultiples of B and F. Conſequently, as A is to
* Def. 5. of this. B, ſo * is E to F. Therefore, *Proportions that are one and the ſame to any third, are alſo the ſame to one another*; which was to be demonſtrated.

PROPOSITION XII.

THEOREM.

If any Number of Magnitudes be proportional, as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents.

LET there be any Number of Proportional Magnitudes, A, B, C, D, E, F; whereof as A is to B,



ſo C is to D, and ſo E to F. I ſay, as A is to B, ſo are all the Antecedents A, C, and E, together, to all the Conſequents, B, D, and F, together.

For, let G, H, and K, be Equimultiples of A, C, and E; and L, M, and N, any other Equimultiples of B, D, and F.

Then, becauſe as A is to B, ſo is C to D, and ſo E to F; and G, H, and K, are Equimultiples of A, C, and E; and L, M, and N, Equimultiples of B, D,
* Def. 5. of this. and F; if G exceeds L, H * will alſo exceed M, and K will exceed N; if G be equal to L, H will be equal to M, and K to N; and if leſs, leſs. Wherefore, alſo, if G exceeds L, then G, H, and K, together, will likewiſe exceed L, M, and N together; and, if G be equal to L, then G, H, and K, together, will be equal to L, M, and N, together; and if leſs, leſs: But G, and G, H, and K, are Equimultiples of A, and A, C, and E; becauſe, if there are any Number of Magnitudes Equimultiples to a like Number of Magnitudes, each to the other, the ſame Multiple that one Magni-
† 1 of this. tude is of one, ſo ſhall † all the Magnitudes be of all.

And, for the ſame Reaſon, L, and L, M, and N, are Equimultiples of B, and B, D, and F. Therefore, as A is to B, ſo * is A, C, and E, together, to B, D, and F, together. Wherefore, *if there be any Number of Magnitudes proportional, as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents*; which was to be demonſtrated.

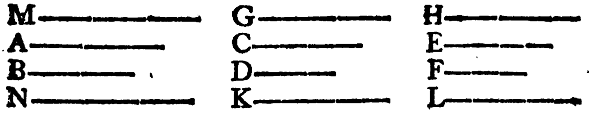
* *Def.* 5. *of this.*

PROPOSITION XIII.

Theorem.

If the firſt has the ſame Proportion to the ſecond, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the ſixth; then alſo ſhall the firſt have a greater Proportion to the ſecond, than the fifth has to the ſixth.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C has to the fourth D; and let the third C have a greater Proportion to the fourth D, than the fifth E to the ſixth F. I ſay, likewiſe, that the firſt A, to the ſecond B, has a greater Proportion, than the fifth E, to the ſixth F.



For, becauſe C has a greater Proportion to D, than E has to F; there are * certain Equimultiples of C and E, and others of D and F, ſuch that the Multiple of C may exceed the Multiple of D; but the Multiple of E not that of F. Now let theſe Equimultiples of C and E be G and H; and K and L thoſe of D and F; ſo that G exceeds K, and H not L: Make M the ſame Multiple of A, as G is of C; and N the ſame of B, as K is of D.

* *Def.* 7. *of this.*

Then, becauſe A is to B, as C is to D; and M and G are Equimultiples of A and C; and N and K of B and D: If M exceeds N, then † G will exceed K; and † if M be equal to N, G will be equal to K; and if leſs,

† *Def.* 5.

leſs. But G does exceed K: Therefore M will alſo exceed N. But H does not exceed L. And M and H are Equimultiples of A and E; and N and L any others of B and F. Therefore A has a * greater Proportion to B, than E has to F. Wherefore, *if the firſt has the ſame Proportion to the ſecond, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the ſixth; then, alſo, ſhall the firſt have a greater Proportion to the ſecond, than the fifth has to the ſixth*; which was to be demonſtrated.

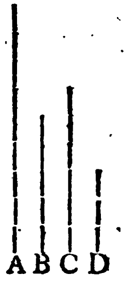
* *Def.* 7. *of this.*

PROPOSITION XIV.

THEOREM.

If the firſt has the ſame Proportion to the ſecond, as the third has to the fourth; and if the firſt be greater than the third; then will the ſecond be greater than the fourth. But, if the firſt be equal to the third, then the ſecond ſhall be equal to the fourth; and, if the firſt be leſs than the third, then the ſecond will be leſs than the fourth.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C has to the fourth D; and let A be greater than C. I ſay, B is alſo greater than D.



For, becauſe A is greater than C, and B is any other Magnitude; A will have a * greater Proportion to B, than C has to B: But as A is to B, ſo is C to D; therefore, alſo, C ſhall ‡ have a greater Proportion to D, than C hath to B. But that Magnitude to which the ſame bears a greater Proportion, is † the leſſer Magnitude. Wherefore D is leſs than B; and conſequently B will be greater than D. In like manner we demonſtrate, if A be equal to C, that B will be equal to D; and if A be leſs than C, that B will be leſs than D. Therefore, *if the firſt has the ſame Proportion to the ſecond, as the third has to the fourth; and if the firſt be greater than the third; then will the ſecond be greater than the fourth. But if the firſt be equal to the third, then*

* 8 *of this.*
‡ 13 *of this.*
† 10 *of this.*

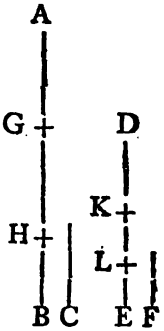
then the ſecond ſhall be equal to the fourth; *and if the firſt be leſs than the third, then the ſecond will be leſs than the fourth*; which was to be demonſtrated.

PROPOSITION XV.

Theorem.

Parts have the ſame Proportion as their like Multiples, if taken correſpondently.

LET A B be the ſame Multiple of C, as D E is of F. I ſay, as C is to F, ſo is A B to D E.



For, becauſe A B and D E are Equimultiples of C and F, there ſhall be as many Magnitudes equal to C in A B, as there are Magnitudes equal to F in D E. Now, let A B be divided into the Magnitudes A G, G H, H B, each equal to C; and E D into the Magnitudes D K, K L, L E, each equal to F; then the Number of the Magnitudes A G, G H, H B, will be equal to the Number of the Magnitudes D K, K L, L E. Now, becauſe A G, G H, H B, are equal, as likewiſe D K, K L, L E; it ſhall be *, as A G is to D K, ſo is G H to K L, and ſo is H B to L E. But as one of the Antecedents is to one of the Conſequents, ſo † all the Antecedents to all the Conſequents. Therefore, as A G is to D K, ſo is A B to D E. But A G is equal to C, and D K to F. Whence, as C is to F, ſo ſhall A B be to D E. Therefore, *Parts have the ſame Proportion as their like Multiples, if taken correſpondently*; which was to be demonſtrated.

* 7 of this.

† 12 of this.

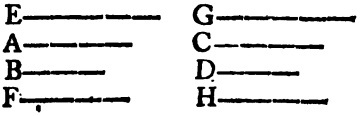
PRO-

PROPOSITION XVI.

THEOREM.

If four Magnitudes of the ſame Kind are proportional, they ſhall alſo be alternately proportional.

LET four Magnitudes A, B, C, D, be proportional; whereof A is to B, as C is to D. I ſay, likewiſe, that they will be alternately proportional; *viz.* as A is to C, ſo is B to D: For take E and F, Equimultiples of A and B; and G and H, any Equimultiples of C and D.



Then, becauſe E is the ſame Multiple of A, as F is of B, and Parts have the ſame Proportion * to their like Multiples, if taken correſpondently; it ſhall be, as A is to B, ſo is E to F. But as A is to B, ſo is C to D. Therefore, alſo, as C is to D, ſo † is E to F. Again, becauſe G and H are Equimultiples of C and D, and Parts have the ſame Proportion with their like Multiples, if taken correſpondently, it will be, as C is to D, ſo is G to H; but as C is to D, ſo is E to F. Therefore, alſo, as E is to F, ſo is G to H; and if four Magnitudes be proportional, and the firſt greater than the third, then the ſecond will be ‡ greater than the fourth; and if the firſt be equal to the third, the ſecond will be equal to the fourth; and if leſs, leſs. Therefore, if E exceeds G, F will exceed H; and if E be equal to G, F will be equal to H; and if leſs, leſs. But E and F are any Equimultiples of A and B; and G and H, any Equimultiples of C and D. Whence as A is to C, ſo ſhall B be * to D. Therefore, *if four Magnitudes of the ſame Kind are proportional, they ſhall alſo be alternately proportional*; which was to be demonſtrated.

* 15 *of this.*
† 11 *of this.*
‡ 14 *of this.*
* *Def.* 5.

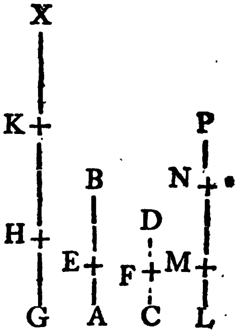
PRO-

PROPOSITION XVII.

Theorem.

If Magnitudes compounded are proportional, they ſhall alſo be proportional when divided.

LET the compounded Magnitudes A B, B E, C D, D F, be proportional; that is, let A B be to B E, as C D is to D F. I ſay, theſe Magnitudes divided are proportional; *viz.* as A E is to E B, ſo is C F to F D.



For let G H, H K, L M, and M N, be Equimultiples of A E, E B, C F, and F D; and K X, and N P, any Equimultiples of E B and F D.

Becauſe G H is the ſame Multiple of A E, as H K is of E B; therefore G H * is the ſame Multiple of A E, as G K is of A B. But G H is the ſame Multiple of A E, as L M is of C F. Wherefore G K is the ſame Multiple of A B, as L M is of C F. Again, becauſe L M is the ſame Multiple of C F, as M N is of F D, L M will be * the ſame Multiple of C F, as L N is of C D. Therefore G K is the ſame Multiple of A B, as L N is of C D. And ſo G K and L N will be Equimultiples of A B and C D. Again, becauſe H K is the ſame Multiple of E B, as M N is of F D; as likewiſe K X the ſame Multiple of E B, as N P is of F D; the compounded Magnitude H X is † alſo the ſame Multiple of E B, as M P is of F D. Wherefore, ſince it is, as A B is to B E, ſo is C D to D F; and G K and L N are Equimultiples of A B and C D; and alſo H X and M P any Equimultiples of E B and F D; if G K exceeds H X, then L N will ‡ exceed M P; and if G K be equal to H X, then L N will be equal to M P; if leſs, leſs. Now let G K exceed H X; then, if H K, which is common, be taken away, G H ſhall exceed K X. But when G K exceeds H X, then L N exceeds M P; therefore L N does exceed M P. If M N, which is common,

* 1 *of this.*

† 2 *of this.*

‡ *Def.* 5.

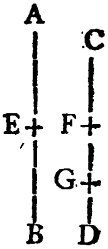
common, be taken away, then L M will exceed N P. And ſo, if G H exceeds K X, then L M will exceed N P. In like manner we demonſtrate, if G H be equal to K X, that L M will be equal to N P; and if leſs, leſs. But G H and L M are Equimultiples of A E and C F; and K X and N P are any Equimul-
* *Def.* 5. tiples of E B and F D. Whence, * as A E is to E B, ſo is C F to F D. Therefore, *if Magnitudes compounded are proportional, they ſhall alſo be proportional when divided*; which was to be demonſtrated.

PROPOSITION XVIII.

Theorem.

If Magnitudes divided be proportional, the ſame alſo being compounded, ſhall be proportional.

LET the divided proportional Magnitudes be A E, E B, C F, F D; that is, as A E is to E B, ſo is C F to F D. I ſay, they are alſo proportional when compounded; *viz.* as A B is to B E, ſo is C D to D F.



For, if A B be not to B E, as C D is to D F, A B ſhall be to B E, as C D is to a Magnitude, either greater or leſs than F D.

Firſt, Let it be to a leſſer, *viz.* to G D. Then, becauſe A B is to B E, as C D is to D G, compounded Magnitudes are pro-
* 17 *of this.* portional; and conſequently * they will be proportional when divided. Therefore A E is to E B, as C G is to G D. But (by the *Hyp.*) as A E is to E B, ſo is C F to F D. Wherefore, alſo, as
† 11 *of this.* C G is to G D, ſo † is C F to F D. But the firſt C G is greater than the third C F; therefore the ſecond
‡ 14 *of this.* D G ſhall be ‡ greater than the fourth D F. But it is leſs, which is abſurd. Therefore A B is not to B E, as C D is to D G. We demonſtrate in the ſame manner, that A B to B E is not as C D to a greater than D F. Therefore A B to B E muſt neceſſarily be as C D is to D F. And ſo, *if Magnitudes divided be proportional, they will alſo be proportional when compounded*; which was to be demonſtrated.

PRO-

PROPOSITION XIX.

Theorem.

If the Whole be to the Whole, as a Part taken away is to a Part taken away; then ſhall the Reſidue be to the Reſidue, as the Whole is to the Whole.

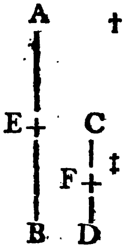
LET the Whole A B be to the Whole C D, as the Part taken away A E, is to the Part taken away C F. I ſay, the Reſidue E B is to the Reſidue F D, as the Whole A B is to the Whole C D.

For, becauſe the Whole A B is to the Whole C D, as A E is to C F; it ſhall be * alternately, as A B is to A E, ſo is C D to C F. Then, becauſe compounded Magnitudes, being proportional, will be † alſo proportional when divided; as B E is to E A, ſo is D F to F C: And again, it will be by * Alternation, as B E is to D F, ſo is E A to F C. But as E A is to F C, ſo (by the *Hyp.*) is A B to C D. And therefore the ‡ Reſidue E B ſhall be to the Reſidue F D, as the Whole A B to the Whole C D. Wherefore, *if the Whole be to the Whole, as a Part taken away is to a Part taken away; then ſhall the Reſidue be to the Reſidue, as the Whole is to the Whole*; which was to be demonſtrated.

* 16 *of this*

† 17 *of this*

‡ 11 *of this*



Coroll. If four Magnitudes be proportional, they will be likewiſe converſly proportional. For let A B be to B E, as C D to D F; then (by Alternation,) it ſhall be, as A B is to C D, ſo is B E to D F. Wherefore, ſince the Whole A B is to the Whole C D, as the Part taken away B E is to the Part taken away D F; the Reſidue A E to the Reſidue C F ſhall be as the Whole A B to the Whole C D. And again, (by Inverſion and Alternation) as A B is to A E, ſo is C D to C F. Which is by converſe Ratio.

The Demonſtration of converſe Ratio, laid down in this Corollary, is only particular. For Alternation (which is uſed herein) cannot be applied but when the four proportional Magnitudes are all of the ſame Kind, as

will appear from the 4th and 17th Definitions of this Book; But converſe Ratio may be uſed when the Terms of the firſt Ratio are not of the ſame Kind with the Terms of the latter. Therefore, inſtead of that, it may not be improper to add this Demonſtration following: If four Magnitudes are proportional, they will be ſo converſly: For let AB *be to*
* 17 of this. BE, *as* CD *to* DF. *And then dividing, it is,* * *as*
† Cor. 4. of this. AE *is to* BE, *ſo is* CF *to* DF: *And this inverſly is,* † *as* BE *is to* AE, *ſo is* DF *to* CF; *which by compound-*
‡ 18 of this. *ing becomes,* ‡ *as* AB *is to* AE, *ſo is* CD *to* CF; *which by the 17th Definition, is converſe Ratio: By* S. Cunn.

PROPOSITION XX.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, being taken two and two in each Order, are in the ſame Ratio; and if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth: But if the firſt be equal to the third, then the fourth will be equal to the ſixth; and if the firſt be leſs than the third, the fourth will be leſs than the ſixth.

LET A, B, C, be three Magnitudes, and D, E, F, others equal to them in Number, which being taken two and two in each Order, are in the ſame Proportion, *viz.* let A be to B, as D is to E; and B to C, as E to F; and let the firſt Magnitude A be greater than the third C. I ſay, the fourth D is alſo greater than the ſixth F. And if A be equal to C, D is equal to F. But if A be leſs than C, D is leſs than F.

For, becauſe A is greater than C, and B is any other Magnitude; and ſince a greater
* 8 of this. Magnitude hath * greater Proportion to the ſame Magnitude than a leſſer hath; A will have a greater Proportion to B, than C
† By Hyp. hath to B. But as A is to B, ſo is D to E †; therefore D hath a greater Proportion to E, than C hath to B. Now inverſly, as C is to B, ſo is F to E. Therefore alſo D will have a greater Propor-

tion to E, than F has to E. But of Magnitudes having Proportion to the ſame Magnitude, that which has the greater Proportion is † the greater Magnitude. † 10 *of this.* Therefore D is greater than F. In the ſame manner we demonſtrate, if A be equal to C, then D will be alſo equal to F; and if A be leſs than C, then D will be leſs than F. Therefore, *if there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the ſame Ratio; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth: But if the firſt be equal to the third, then the fourth will be equal to the ſixth; and if the firſt be leſs than the third, the fourth will be leſs than the ſixth*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which taken two and two, are in the ſame Proportion, and the Proportion be perturbate; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth; but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, leſs.

LET three Magnitudes, A, B, C, be proportional; and others, D, E, F, equal to them in Number. Let their Analogy likewiſe be perturbate; *viz.* as A is to B, ſo is E to F; and as B is to C, ſo is D to E: If the firſt Magnitude A be greater than the third C, I ſay, the fourth D is alſo greater than the ſixth F. And if A be equal to C, then D is equal to F; but if A be leſs than C, then D is leſs than F.

For, ſince A is greater than C, and B is ſome other Magnitude, A will have * a greater Proportion to B, than C has to B. * 8 *of this.* But as A is to B, ſo is E to F; whence E has a greater Proportion to F, than C hath to B: Now inverſly, as C is to B, ſo is E to D: Wherefore alſo, E ſhall have a greater Proportion to F, than E to D. But that Magnitude to which the ſame Magnitude

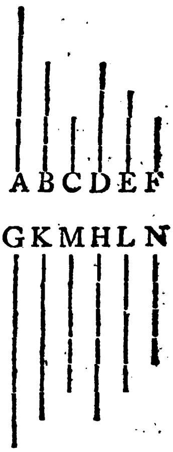
† 10 *of this.* bears a greater Proportion, † is the leſſer Magnitude, Therefore F is leſs than D; and ſo D ſhall be greater than F. After the ſame manner we demonſtrate, if A be equal to C, D will be alſo equal to F; and if A be leſs than C, D will alſo be leſs than F. *If*, therefore, *there are three Magnitudes, and others equal to them in Number, which, taken two and two, are in the ſame Proportion, and the Proportion be perturbate; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth; but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, leſs;* which was to be demonſtrated.

PROPOSITION XXII.

THEOREM.

If there be any Number of Magnitudes and others equal to them in Number, which, taken two and two, are in the ſame Proportion; then they ſhall be in the ſame Proportion by Equality.

LET there be any Number of Magnitudes, A, B, and C; and others, D, E, and F, equal to them in Number, which, taken two and two, are in the ſame Proportion; that is, as A is to B, ſo is D to E; and as B is to C, ſo is E to F. I ſay, they are alſo proportional by Equality, *viz.* as A is to C, ſo is D to F.



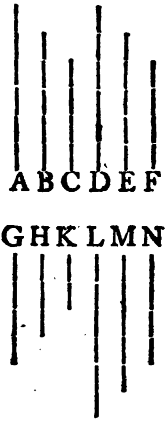
For let G and H be any Equimultiples of A and D; and K and L any Equimultiples of B and E; and likewiſe M and N, any Equimultiples of C and F. Then, becauſe A is to B, as D is to E; and G and H are Equimultiples of A and D; and K and L Equimultiples of B and
* 4 *of this.* E, it ſhall be, * as G is to K, ſo is H to L. For the ſame Reaſon, alſo, it will be as K is to M, ſo is L to N. And ſince there are three Magnitudes G, K, and M, and others H, L, and N, equal to them in Number, which being taken two and two, in each

Order, are in the ſame Proportion; therefore if G exceeds M, * H will exceed N; if G be equal to M, * 20 *of this.*
then H ſhall be equal to N; and if G be leſs than M, H ſhall be leſs than N. But G and H are Equimultiples of A and D; and M and N, any other Equimultiples of C and F. Whence as A is to C, ſo ſhall D † be to F. Therefore, *if there be any Number of* † *Def.* 5. *of this.*
Magnitudes, and others equal to them in Number, which taken two and two, are in the ſame Proportion; then they ſhall be in the ſame Proportion by Equality; which was to be demonſtrated.

PROPOSITION XXIII.

Problem.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the ſame Proportion; and if their Analogy be perturbate, then ſhall they be alſo in the ſame Proportion by Equality.



LET there be three Magnitudes A, B, and C; and others equal to them in Number, D, E, and F, which, taken two and two, are in the ſame Proportion, and their Analogy be perturbate; that is, as A is to B, ſo is E to F; and as B is to C, ſo is D to E. I ſay, as A is to C, ſo is D to F.

For, let G, H, and L, be Equimultiples of A, B, and D; and K, M, and N, any Equimultiples of C, E, and F.

Then, becauſe G and H are Equimultiples of A and B, and ſince Parts have the ſame Proportion as their like Multiples, when taken correſpondently; it ſhall be *, as A is to B, ſo is G to H; * 15 *of this.*
and, by the ſame Reaſon, as E is to F, ſo is M to N. But A is to B, as E to F. Therefore, ‡ as G is to ‡ 11 *of this.*
H, ſo is M to N. Again, becauſe B is to C as D is to E; and H and L are Equimultiples of B and D;

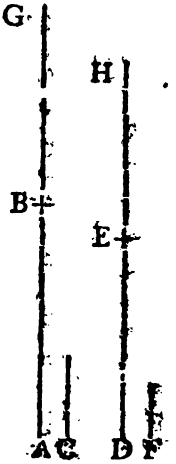
as likewiſe K and M any Equimultiples of C and E; it ſhall be, as H is to K, ſo is L to M. But it has been alſo proved, that as G is to H, ſo is M to N. Therefore, becauſe three Magnitudes G, H, and K, and others, L, M, and N, equal to them in Number, which, taken two and two, are in the ſame Proportion, and their Analogy is perturbate; then if G exceeds K, alſo L * will exceed N; and if G be equal to K, then L will be equal to N; and if G be leſs than K, L will likewiſe be leſs than N. But G and L are Equimultiples of A and D; and K and N Equimultiples of C and F. Therefore, as A is to C, ſo ſhall D be to F. Wherefore, *if there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the ſame Proportion; and if their Analogy be perturbate, then ſhall they be alſo in the ſame Proportion by Equality*; which was to be demonſtrated.

* 21 *of this.*

PROPOSITION XXIV.

THEOREM.

If the firſt Magnitude has the ſame Proportion to the ſecond, as the third to the fourth; and if the fifth has the ſame Proportion to the ſecond, as the ſixth has to the fourth; then ſhall the firſt compounded with the fifth, have the ſame Proportion to the ſecond, as the third, compounded with the ſixth, has to the fourth.



LET the firſt Magnitude AB have the ſame Proportion to the ſecond C, as the third DE has to the fourth F. Let alſo the fifth BG have the ſame Proportion to the ſecond C, as the ſixth EH has to the fourth F. I ſay, AG, the firſt compounded with the fifth has the ſame Proportion to the ſecond C, as DH, the third compounded with the ſixth, has to the fourth F.

For, becauſe B G is to C, as E H is to F; it ſhall be (inverſly), as C is to B G, ſo is F to E H. Then, ſince A B is to C, as D E is to F; and as C is to B G, ſo is F to E H; it ſhall be, * by Equality, as A B is to B G, ſo is D E to E H. And becauſe Magnitudes, being divided, are proportional, they ſhall alſo be † proportional when compounded. Therefore, as A G is to G B, ſo is D H to H E: But as G B is ‡ to C, ſo alſo is H E to F. Wherefore, by Equality *, it ſhall be, as A G is to C, ſo is D H to F. Therefore, *if the firſt Magnitude has the ſame Proportion to the ſecond, as the third to the fourth; and if the fifth has the ſame Proportion to the ſecond, as the ſixth has to the fourth; then ſhall the firſt, compounded with the fifth, have the ſame Proportion to the ſecond, as the third, compounded with the ſixth, has to the fourth*; which was to be demonſtrated.

* 22 *of this.*

† 18 *of this.*

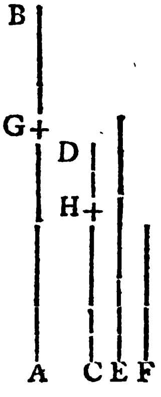
‡ *Hyp.*

PROPOSITION XXV.

THEOREM.

If four Magnitudes be proportional; the greateſt, and the leaſt of them, will be greater than the other two.

LET four Magnitudes, A B, C D, E, F, be proportional, whereof A B is to C D, as E is to F; let A B be the greateſt of them, and F the leaſt. I ſay, A B and F, are greater than C D and E.



For, let A G be equal to E, and C H to F. Then, becauſe A B is to C D as E is to F; and ſince A G and C H are each equal to E and F; it ſhall be as A B is to D C, ſo is A G to C H. And becauſe the Whole A B is to the Whole C D, as the Part taken away A G is to the Part taken away C H; it ſhall alſo be *, as the Reſidue G B to the Reſidue H D, ſo is the Whole A B to the Whole C D. But A B is greater than C D; therefore, alſo,

* 19 *of this.*

G B ſhall be greater than H D. And ſince A G is equal to E, and C H to F; A G and F will be equal to C H and E. But if equal Things are added to unequal Things, the Wholes ſhall be unequal. Therefore G B, H D, being unequal, for G B is the greater, if A G and F are added to G B; and C H and E to H D; then A B and F will neceſſarily be greater than C D and E. Wherefore, *if four Magnitudes be proportional; the greateſt, and the leaſt of them will be greater than the other two*; which was to be demonſtrated.

The END *of the* FIFTH BOOK.

EUCLID's ELEMENTS.

BOOK VI.

DEFINITIONS.

I. *SIMILAR Right-lined Figures are ſuch as have each of their ſeveral Angles equal to one another, and the Sides about the equal Angles proportional to each other.*

II. *Figures are ſaid to be reciprocal, when the antecedent and conſequent Terms of the Ratios are in each Figure.*

III. *A Right Line is ſaid to be cut into mean and extreme Proportion, when the Whole is to the greater Segment, as the greater Segment is to the leſſer.*

IV. *The Altitude of any Figure is a perpendicular Line drawn from the Top, or Vertex, to the Baſe.*

V. *A Ratio is ſaid to be compounded of Ratios, when the Quantities of the Ratios, being multiplied into one another, do produce a Ratio.*

PROPOSITION I.

THEOREM.

Triangles and Parallelograms, that have the ſame Altitude, are to each other as their Baſes.

LET the Triangles A B C, A C D, and the Parallelograms E C, C F, have the ſame Altitude, *viz.* the Perpendicular drawn from the Point A to B D. I ſay, as the Baſe B C is to the Baſe C D, ſo is the Triangle A B C, to the Triangle A C D; and ſo is the Parallelogram E C to the Parallelogram C F.

For, produce B D both Ways to the Points H and L; and take G B, G H, any Number of Times equal to the Baſe B C; and D K, K L, any Number of Times equal to the Baſe C D; and join A H, A G, A K, A L.

Then, becauſe C B, B G, G H, are equal to one another, the Triangles A H G, A G B, A B C, alſo will be* equal to one another: Therefore the ſame Multiple that the Baſe H C is of the Baſe B C, ſhall the Triangle A H C be of the Triangle A B C. By the ſame Reaſon, the ſame Multiple that the Baſe L C is of the Baſe C D, ſhall the Triangle A L C be of the Triangle A C D. And if the Baſe H C be equal to the Baſe C L, the Triangle A H C is alſo * equal to the Triangle A L C: And if the Baſe H C exceeds the Baſe C L, then the Triangle A H C will exceed the Triangle A L C. And if the Baſe H C be leſs than C L, then the Triangle A H C will be leſs than A L C. Therefore, ſince there are four Magnitudes, *viz.* the two Baſes B C, C D, and the two Triangles A B C, A C D; and ſince the Baſe H C, and the Triangle A H C, are Equimultiples of the Baſe B C, and the Triangle A B C: And the Baſe C L, and the Triangle A L C, are Equimultiples of the Baſe C D, and the Triangle A D C: And it has been proved, that if the Baſe H C exceeds the Baſe C L, the Triangle A H C will exceed the Triangle A L C; and if equal, equal; if leſs, leſs: Therefore, as the Baſe B C is to the Baſe C D, ſo † is the Triangle A B C to the Triangle A C D.

* 38. 1.

† Def. 5. 5.

And becauſe the Parallelogram E C is † double to † 41. 1. the Triangle A B C; and the Parallelogram F C, double † to the Triangle A C D; and Parts have the ſame Proportion * as their like Multiples: There- * 15. 5. fore, as the Triangle A B C is to the Triangle A C D, ſo is the Parallelogram E C to the Parallelogram C F. And ſo, ſince it has been proved that the Baſe B C is to the Baſe C D, as the Triangle A B C is to the Triangle A C D; and the Triangle A B C is to the Triangle A C D, as the Parallelogram E C is to the Parallelogram C F; it ſhall be ‡, as the Baſe B C is ‡ 11. 5. to the Baſe C D, ſo is the Parallelogram E C to the Parallelogram F C. Wherefore, *Triangles and Parallelograms, that have the ſame Altitude, are to each other as their Baſes*; which was to be demonſtrated.

PROPOSITION II.

Theorem.

If a Right Line be drawn parallel to one of the Sides of a Triangle, it ſhall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line, joining the Points of Section, ſhall be parallel to the other Side of the Triangle.

LET D E be drawn parallel to B C, a Side of the Triangle A B C. I ſay, D B is to D A, as C E is to E A.

For, let B E, C D be joined.

Then the Triangle B D E is * equal to the Tri- * 37. 1. angle C D E; for they ſtand upon the ſame Baſe D E, and are between the ſame Parallels D E and B C, and A D E is ſome other Triangle. But equal Magnitudes have † the ſame Proportion to one and † 7. 5. the ſame Magnitude. Therefore, as the Triangle B D E is to the Triangle A D E, ſo is the Triangle C D E to the Triangle A D E.

But as the Triangle B D E is to the Triangle ‡ 1 *of this.* A D E, ſo ‡ is B D to D A; for ſince they have the ſame Altitude, *viz.* a Perpendicular drawn from the Point E to A B, they are to each other as their Baſes. And, for the ſame Reaſon, as the Triangle C D E is to the Triangle A D E, ſo is C E to E A: And there- * 11. 3. fore as B D is to D A, ſo is * C E to E A.

And if the Sides A B, A C, of the Triangle A B C, be cut proportionally; that is, ſo that B D be to D A, as C E is to E A; and if D E be joined; I ſay, D E is parallel to B C.

For, the ſame Conſtruction remaining, becauſe B D † 1 *of this.* is to D A, as C E is to E A; and B D is † to D A, as the Triangle B D E is to the Triangle A D E; and C E is to E A, as the Triangle C D E is to the Triangle A D E; it ſhall be as the Triangle B D E is to the Triangle A D E, ſo is * the Triangle C D E to the Triangle A D E. And ſince the Triangle B D E, C D E, have the ſame Proportion to the Triangle † 9. 5. A D E, the Triangle B D E ſhall be † equal to the Triangle C D E; and they have the ſame Baſe D E: ‡ 39. 1. But equal Triangles, being upon the ſame Baſe, ‡ are between the ſame Parallels; therefore D E is parallel to B C. Wherefore, *if a Right Line be drawn parallel to one of the Sides of a Triangle, it ſhall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line, joining the Points of Section, ſhall be parallel to the other Side of the Triangle;* which was to be demonſtrated.

PROPOSITION III.

Theorem.

If one Angle of a Triangle be biſected, and the Right Line, that biſects the Angle, cuts the Baſe alſo; then the Segments of the Baſe will have the ſame Proportion as the other Sides of the Triangle. And if the Segments of the Baſe have the ſame Proportion that the other Sides of the Triangle have; then a Right Line, drawn from the Vertex, to the Point of Section of the Baſe, will biſect the Angle of the Triangle.

LET there be a Triangle A B C, and let its Angle * 9. 1. B A C be * biſected by the Right Line A D. I ſay, as B D is to D C, ſo is B A to A C.

For, thro' C draw * C E parallel to D A, and produce B A, till it meets C E in the Point E. * 31. 1.

Then, becauſe the Right Line A C falls on the Parallels A D, E C, the Angle A C E will be ‡ equal ‡ 29. 1. to the Angle C A D: But the Angle C A D (by the *Hypotheſis*) is equal to the Angle B A D. Therefore the Angle B A D will be equal to the Angle A C E. Again, becauſe the Right Line B A E falls on the Parallels A D, E C, the outward Angle B A D is † equal † 7. 5. to the inward Angle A E C; but the Angle A C E has been proved equal to the Angle B A D: Therefore A C E ſhall be equal to A E C; and ſo the Side A E is equal ‡ to the Side A C. And becauſe the ‡ 6. [illegible] Line A D is drawn parallel to C E, the Side of the Triangle B C E, it ſhall be, * as B D is to D C, ſo is * 2 of this. B A to A E; but A E is equal to A C. Therefore, as B D is to D C, ſo is † B A to A C.

And if B D be to D C, as B A is to A C; and the Right Line A D be joined; then, I ſay, the Angle B A C is biſected by the Right Line A D.

For, the ſame Conſtruction remaining, becauſe B D is to D C, as B A is to A C; and as B D is to D C, ſo is ‡ B A to A E; for A D is drawn parallel to one ‡ 2 of this. Side E C of the Triangle B C E; it ſhall be, as B A is to A C, ſo is B A to A E. Therefore A C is equal to A E †; and, accordingly, the Angle A E C is equal † 9. 4. to the Angle E C A: But the Angle A E C is equal * to the outward Angle B A D; and the Angle * 29. 1. A C E equal to the alternate Angle C A D. Wherefore the Angle B A D is alſo equal to the Angle C A D; and ſo the Angle B A C is biſected by the Right Line A D. Therefore, *if one Angle of a Triangle be biſected, and the Right Line, that biſects the Angle, cuts the Baſe alſo; then the Segments of the Baſe will have the ſame Proportion as the other Sides of the Triangle. And if the Segments of the Baſe have the ſame Proportion that the other Sides of the Triangle have; then a Right Line, drawn from the Vertex, to the Point of Section of the Baſe, will biſect the Angle of the Triangle*; which was to be demonſtrated.

PRO-

PROPOSITION IV.

Theorem.

The Sides about the equal Angles of equiangular Triangles are proportional; and the Sides, which are ſubtended under the equal Angles, are homologous, or of like Ratio.

LET ABC, DEC, be equiangular Triangles, having the Angle ABC equal to the Angle DCE, the Angle ACB equal to the Angle DEC, and the Angle BAC equal to the Angle CDE. I ſay, the Sides that are about the equal Angles of the Triangles ABC, DCE, are proportional; and the Sides that are ſubtended under the equal Angles, are homologous, or of like Ratio.

Set the Side BC in the ſame Right Line with the
Side CE; and becauſe the Angles ABC, ACB, are
* 17. 1. * leſs than two Right Angles, and the Angle ACB
is equal to the Angle DEC, the Angles ABC,
DEC, are leſs than two Right Angles. And ſo
† Ax. 12. BA, ED, produced, will meet † each other; let
them be produced, and meet in the Point F. Then,
becauſe the Angle DCE is equal to the Angle
‡ 28. 1. ABC, BF ſhall be ‡ parallel to DC. Again, be-
cauſe the Angle ACB is equal to the Angle DEC,
the Side AC will be ‡ parallel to the Side FE;
therefore FACD is a Parallelogram, and conſe-
* 34. 1. quently FA is * equal to DC, and AC to FD; and
becauſe AC is drawn parallel to FE, the Side of the
† 2 *of this.* Triangle FBE, it ſhall † be, as BA is to AF, ſo is
BC to CE: But CD is equal to AF, and (by Al-
ternation) as BA is to BC, ſo is CD to CE. Again,
becauſe CD is parallel to BF, it ſhall be † as BC is
to CE, ſo is FD to DE, but FD is equal to AC.
‡ 7. 5. Therefore as BC is to CE, ſo is ‡ AC to DE:
And ſo by Alternation, as BC is to CA, ſo is CE to
ED. Wherefore, becauſe it is demonſtrated, that
AB is to BC, as DC is to CE; and as BC is to
* 21. 5. CA, ſo is CE to ED; it ſhall be, * by Equality, as
BA is to AC, ſo is CD to DE. Therefore, *the
Sides about the equal Angles of equiangular Triangles are
proportional; and the Sides, which are ſubtended under*

the equal Angles, are homologous, or of like Ratio; which was to be demonſtrated.

PROPOSITION V.

Theorem.

If the Sides of two Triangles are proportional, the Triangles ſhall be equiangular; and their Angles, under which the homologous Sides are ſubtended, are equal.

LET there be two Triangles A B C, D E F, having their Sides proportional; that is, let A B be to B C, as D E is to E F; and as B C to C A, ſo is E F to F D: And, alſo, as B A to C A, ſo E D to D F. I ſay, the Triangle A B C is equiangular to the Triangle D E F; and the Angles are equal, under which the homologous Sides are ſubtended, *viz.* the Angle A B C equal to the Angle D E F; and the Angle B C A equal to the Angle E F D; and the Angle B A C equal to the Angle E D F.

For, at the Points E and F, with the Line E F, make * the Angle F E G equal to the Angle A B C; * 23. 1.
and the Angle E F G equal to the Angle B C A: Then the remaining Angle B A C is † is equal to the † *Cor.* 32. 1.
remaining Angle E G F.

And ſo the Triangle A B C is equiangular to the Triangle E G F; and, conſequently, the Sides that are ſubtended under the equal Angles, are proportional. Therefore, as A B is to B C, ſo is ‡ G E to ‡ 4 *of this.*
E F; but (by the *Hyp.*) as A B is to B C, ſo is D E to E F: Therefore, as D E is to E F, ſo is * G E to * 11. 5.
E F. And ſince D E, E G, have the ſame Proportion to E F, D E ſhall be † equal to E G. For the † 9. 5.
ſame Reaſon D F is equal to F G; but E F is common. Then, becauſe the two Sides D E, E F, are equal to the two Sides G E, E F, and the Baſe D F is equal to the Baſe F G, the Angle D E F is ‡ equal to ‡ 8. 1.
the Angle G E F; and the Triangle D E F equal to the Triangle G E F; and the other Angles of the one equal to the other Angles of the other, which are ſubtended by the equal Sides. Therefore the Angle D F E is equal to the Angle G F E, and the Angle E D F equal to the Angle E G F. And becauſe the

Angle DEF is equal to the Angle GEF; and the Angle GEF equal to the Angle ABC; therefore the Angle ABC ſhall be alſo equal to the Angle DEF. For the ſame Reaſon, the Angle ACB ſhall be equal to the Angle DFE; as alſo the Angle A equal to the Angle D: Therefore the Triangle ABC will be equiangular to the Triangle DEF. Wherefore, *if the Sides of two Triangles are proportional, the Triangles ſhall be equiangular; and their Angles, under which the homologous Sides are ſubtended, are equal*; which was to be demonſtrated.

PROPOSITION VI.

Theorem.

If two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular; and have thoſe Angles equal, under which are ſubtended the homologous Sides.

LET there be two Triangles ABC, DEF, having one Angle BAC, of the one, equal to the Angle EDF of the other; and let the Sides about the equal Angles be proportional, *viz.* let AB be to AC, as ED is to DF. I ſay, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC equal to the Angle DEF; and the Angle ACB equal to the Angle DFE.

For, at the Points D and F, with the Right Line
* 23. 1. DF, make * the Angle FDG equal to either of the
Angles BAC, EDF; and the Angle DFG equal
to the Angle ACB.

† Cor. 32. 1. Then the other Angle B is † equal to the other
Angle G; and ſo the Triangle ABC is equiangular
to the Triangle DGF; and, conſequently, as BA is
‡ 4 of this. to AC, ſo is ‡ GD to DF: But (by the *Hyp.*) as
BA is to AC, ſo is ED to DF. Therefore, as
* 11. 5. ED is * to DF, ſo is GD to DF; whence ED is
† 9. 5. † equal to DG, and DF is common; therefore the
two Sides ED, DF, are equal to the two Sides GD,
DF; and the Angle EDF equal to the Angle
GDF:

G D F: Conſequently the Baſe E F is * equal to the * 4. 1. Baſe F G, and the Triangle D E F equal to the Triangle D G F; and the other Angles of the one equal to the other Angles of the other, each to each; under which the equal Sides are ſubtended. Therefore the Angle D F G is equal to the Angle D F E, and the Angle G equal to the Angle E; but the Angle D F G is equal to the Angle A C B: Wherefore the Angle A C B is equal to the Angle D F E; but the Angle B A C is ‡ alſo equal to the Angle E D F: ‡ *By Hyp.* Therefore the other Angle at B is † equal to the other † 32. 1. Angle at E; and ſo the Triangle A B C is equiangular to the Triangle D E F. Therefore, *if two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional; then the Triangles are equiangular; and have thoſe Angles equal, under which are ſubtended the homologous Sides*; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If there are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about the other Angles proportional; and if the remaining third Angles are either both leſs, or both not leſs, than Right Angles; then ſhall the Triangles be equiangular, and have thoſe Angles equal, about which are the proportional Sides.

LET two Triangles A B C, D E F, have one Angle of the one equal to one Angle of the other, *viz.* the Angle B A C equal to the Angle E D F; and let the Sides about the other Angles A B C, D E F, be proportional, *viz.* as D E is to E F, ſo let A B be to B C; and let the other Angles at C and F be both leſs, or both not leſs, than Right Angles. I ſay, the Triangle A B C is equiangular to the Triangle D E F; and the Angle A B C is equal to the Angle D E F; as alſo the other Angle at C equal to the other Angle at F.

For, if the Angle A B C be not equal to the Angle
D E F, one of them will be the greater, which let be
A B C. Then at the Point B with the Right Line
* 23. 1. A B, make * the Angle A B G equal to the Angle
D E F.

Now, becauſe the Angle A is equal to the Angle
D, and the Angle A B G equal to the Angle D E F;
† Cor. 32. 1. the remaining Angle A G B is † equal to the remaining Angle D F E: And therefore the Triangle A B G
is equiangular to the Triangle D E F; and ſo, as A B
‡ 4 of this. is to B G, ſo is ‡ D E to E F; but as D E is to E F,
* By Hyp. ſo is * A B to B C. Therefore, as A B is to B C, ſo
‡ 11. 5. is A B to ‡ B G; and ſince A B has the ſame Propor-
† 9. 5. tion to B C, that it has to B G, B C ſhall be † equal
* 5. 1. to B G; and, conſequently, the Angle at C * equal
to the Angle B G C. Wherefore each of the Angles B C G, or B G C, is leſs than a Right Angle; and conſequently A G B is greater than a Right Angle. But the Angle A G B has been proved equal to the Angle at F; therefore the Angle at F is greater than a Right Angle: But (by the *Hyp.*) it is not greater, ſince C is not greater than a Right Angle, which is abſurd. Wherefore the Angle A B C is not unequal to the Angle D E F; and ſo it muſt be equal to the ſame; but the Angle at A is equal to that at D; wherefore the Angle remaining at C is equal to the remaining Angle at F; and, conſequently, the Triangle A B C is equiangular to the Triangle D E F. Therefore, *if there are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about the other Angles proportional; and if the remaining third Angles are either both leſs, or both not leſs, than Right Angles; then ſhall the Triangles be equiangular, and have thoſe Angles equal, about which are the proportional Sides*; which was to be demonſtrated.

PRO-

PROPOSITION VIII.

Theorem.

If a Perpendicular be drawn, in a Right-angled Triangle, from the Right Angle to the Baſe, then the Triangle on each Side of the Perpendicular are ſimilar, both to the Whole, and alſo to one another.

LET A B C be a Right-angled Triangle, whoſe Right Angle is B A C; and let the Perpendicular A D, be drawn from the Point A to the Baſe B C. I ſay, the Triangles A B D, A D C, are ſimilar to one another, and to the whole Triangle A B C.

For, becauſe the Angle B A C is equal to the Angle A D B, for each of them is a Right Angle; and the Angle at B is common to the two Triangles A B C, A B D; the remaining Angle A C B ſhall be * equal to the remaining Angle B A D. Therefore * *Cor.* 32. 1. the Triangle A B C is equiangular to the Triangle A B D; and ſo, as † B C, which ſubtends the Right † 4 *of this.* Angle of the Triangle A B C, is to B A, ſubtending the Right Angle of the Triangle A B D, ſo is A B, ſubtending the Angle C of the Triangle A B C, to D B, ſubtending an Angle equal to the Angle C, *viz.* the Angle B A D, of the Triangle A B D; and ſo, moreover, is A C to A D, ſubtending the Angle B, which is common to the two Triangles. Therefore the Triangle A B C is ‡ equiangular to the Triangle ‡ *Def.* 1. *of this.* A B D; and the Sides about the equal Angles are proportional. Wherefore the Triangle A B C is ‡ ſimilar to the Triangle A B D. By the ſame Way we demonſtrate, that the Triangle A D C is alſo ſimilar to the Triangle A B C. Wherefore each of the Triangles A B D, A D C, is ſimilar to the whole Triangle.

I ſay, the ſaid Triangles are alſo ſimilar to one another.

For, becauſe the Right Angle B D A is equal to the Right Angle A D C, and the Angle B A D has been proved equal to the Angle C; it follows, that the remaining Angle at B * ſhall be equal to the remaining * *Cor.* 32. 1.

Angle D A C. And ſo the Triangle A B D is equi- † 4 of this. angular to the Triangle A D C. Wherefore as † B D, ſubtending the Angle B A D of the Triangle A B D, is to D A, ſubtending the Angle at C of the Triangle ADC, which is equal to the Angle B A D; ſo is A D, ſubtending the Angle B of the Triangle A B D, to D C, ſubtending the Angle D A C, equal to the Angle B. And, moreover, ſo is B A to A C, ſubtending the Right Angles at D; and, conſequently, the Triangle A B D is ſimilar to the Triangle A D C. Wherefore, *if a Perpendicular be drawn in a Right-angled Triangle, from the Right Angle to the Baſe, then the Triangles on each Side of the Perpendicular are ſimilar, both to the Whole, and alſo to one another*; which was to be demonſtrated.

Coroll. From hence it is manifeſt, that the Perpendicular drawn in a Right-angled Triangle from the Right Angle to the Baſe, is a mean Proportional between the Segments of the Baſe. Moreover, either of the Sides containing a Right Angle, is a mean Proportional between the whole Baſe, and the Segment thereof, which is next to the Side.

PROPOSITION IX.

PROBLEM.

To cut off any Part required from a given Right Line.

LET A B be a Right Line given, from which muſt be cut off any required Part; ſuppoſe a third.

Draw any Right Line A C from the Point A, making an Angle at Pleaſure with the Line A B. Aſſume * 3. 1. any Part D in the Line A C; make * D E, E C, each † 31. 1. equal to A D; join B C, and draw † D F thro' D, parallel to B C.

Then, becauſe F D is drawn parallel to the Side ‡ 2 of this. B C of the Triangle A B C, it ſhall be, ‡ as C D is to D A, ſo is B F to F A. But C D is double to D A. Therefore B F ſhall be double to F A; and ſo B A is triple to A F. Wherefore, *there is cut off* A F, *a third Part required, of the given Right Line* A B; which was to be done.

PROPOSITION X.

Problem.

To divide a given undivided Right Line, as another given Right Line is divided.

LET A B be a given undivided Right Line, and A C a divided Line. It is required to divide A B, as A C is divided.

Let A C be divided in the Points D and E, and ſo placed, as to contain any Angle with A B. Join the Points C and B; through D and E let D F, E G, be drawn * parallel to B C; and through D draw * 31. 1. D H K, parallel to A B.

Then F H, H B, are each of them Parallelograms; and ſo D H is † equal to F G, and H K to G B. And † 34. 1. becauſe H E is drawn parallel to the Side K C of the Triangle D K C, it ſhall be ‡ as C E is to E D, ſo is ‡ 2 *of this.* K H to H D. But K H is equal to B G, and H D to G F. Therefore, as C E is to E D, ſo is B G to G F. Again, becauſe F D is drawn parallel to the Side E G of the Triangle A G E, as E D is to D A, ſo ſhall † G F be to F A. But it has been proved, that C E is to E D, as B G is to G F. Therefore, as C E is to E D, ſo is B G to G F; and as E D is to D A, ſo is G F to F A. Wherefore, *the given undivided Line* A B *is divided as the given Line* A C *is*; which was to be done.

PROPOSITION XI.

Problem.

Two Right Lines being given, to find a third Proportional to them.

LET A B, A C, be two given Right Lines, ſo placed, as to make any Angle with each other. It is required to find a third Proportional to A B, A C.

Produce A B, A C, to the Points D and E; make B D equal to A C; join the Points B, C; and draw ‡ the Right Line D E thro' D, parallel to B C. ‡ 31. 1.

Then, becauſe B C is drawn parallel to the Side D E of the Triangle A D E, it ſhall be, ‡ as A B is to B D, ſo is A C to C E. But B D is equal to A C. Hence, as A B is to A C, ſo is A C to C E. Therefore, *a third Proportional* C E *is found to two given Right Lines* A B, A C; which was to be done.

‡ 2 *of this.*

PROPOSITION XII.

PROBLEM.

Three Right Lines being given, to find a fourth Proportional to them.

LET A, B, C, be three Right Lines given. It is required to find a fourth Proportional to them.

Let D E and D F be two Right Lines, making any Angle E D F with each other. Now make D G equal to A, G E equal to B, D H equal to C; and draw the Line G H, as alſo, † E F thro' E, parallel to G H.

† 31. 1.

Then, becauſe G H is drawn parallel to E F, the Side of the Triangle D E F, it ſhall be, as D G is to G E, ſo is D H to H F. But D G is equal to A, G E to B, and D H to C. Conſequently, as A is to B, ſo is C to H F. Therefore, *the Right Line* H F, *a fourth Proportional to the three given Right Lines* A, B, C, *is found*; which was to be done.

PROPOSITION XIII.

PROBLEM.

To find a mean Proportional between two given Right Lines.

LET the two given Right Lines be A B, B C. It is required to find a mean Proportional between them. Place A B, B C, in a direct Line; and on the Whole A C deſcribe the Semicircle A D C, and ‡ draw B D at Right Angles to A C from the Point B; and let A D, D C, be joined.

‡ 11. 1.

Then, becauſe the Angle A D C, in a Semicircle, is † a Right Angle; and ſince the Perpendicular D B is drawn from the Right Angle to the Baſe; therefore D B

† 31. 3.

D B is ‡ a mean Proportional between the Segments of the Baſe A B, B C. Wherefore, *a mean Proportional between the two given Lines* A B, B C, *is found*; which was to be done.

‡ Cor. 8. of this.

PROPOSITION XIV.

Theorem.

Equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and thoſe Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal.

LET A B, B C, be equal Parallelograms, having the Angles at B equal; and let the Sides D B, B E, be in one ſtrait Line; then alſo will ‡ the Sides F B, B G, be in one ſtrait Line. I ſay, the Sides of the Parallelograms A B, B C, that are about the equal Angles, are reciprocal; that is, as D B is to B E, ſo is G B to B F.

‡ 14. 1.

For, let the Parallelogram F E be compleated.

Then, becauſe the Parallelogram A B is equal to the Parallelogram B C, and F E is ſome other Parallelogram; it ſhall be, as A B is to F E, ſo is † B C to F E; but as A B is to F E, ſo is ‡ D B to B E; ‡ and as B C is to F E, ſo is G B to B F. Therefore as D B is to B E, ſo is G B to B F. Wherefore the Sides of the Parallelograms A B, B C, that are about the equal Angles, are reciprocally proportional.

† 7. 5.

‡ 1 of this.

And if the Sides that are about the equal Angles are reciprocally proportional, *viz.* if D B be to B E, as G B is to B F; I ſay, the Parallelogram A B is equal to the Parallelogram B C.

For, ſince D B is to B E, as G B is to B F; and D B to B E, as the Parallelogram A B ‡ to the Parallelogram F E; and G B ‡ to B F, as the Parallelogram B C to the Parallelogram F E; it ſhall be as A B is to F E, ſo is B C † to F E. Therefore the Parallelogram A B is equal to the Parallelogram B C. And ſo, *equal Parallelograms, having one Angle of the* *one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and thoſe Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal;* which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

Equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and thoſe Triangles that have one Angle of the one equal to one Angle of the other, and have alſo the Sides about the equal Angles reciprocal, are equal.

LET the equal Triangles A B C, A D E, have one Angle of the one equal to one Angle of the other, *viz.* the Angle B A C equal to the Angle D A E. I ſay, the Sides about the equal Angles are reciprocal; that is, as C A is to A D, ſo is E A to A B.

For, place C A and A D in one ſtrait Line; then ‡ 14. 1. E A and A B ſhall be ‡ alſo in one ſtrait Line; and let B D be joined. Then, becauſe the Triangle A B C is equal to the Triangle A D E, and A B D is ſome † 7. 5. other Triangle, the Triangle C A B ſhall be † to the Triangle B A D, as the Triangle A D E is to the Triangle B A D. But, as the Triangle C A B is to ‡ 1 *of this.* the Triangle B A D, ſo is C A ‡ to A D; and as the Triangle E A D is to the Triangle B A D, ſo ‡ is E A * 11. 5. to A B. Therefore, as C A is to A D, * ſo is E A to A B. Wherefore the Sides of the Triangles A B C, A D E, about the equal Angles are reciprocal.

And, if the Sides about the equal Angles of the Triangles A B C, A D E, be reciprocal, *viz.* if C A be to A D, as E A is to A B; I ſay, the Triangle A B C is equal to the Triangle A D E,

For, again, let B D be joined. Then, becauſe C A is to A D, as E A is to A B; and C A to A D ‡, as the Triangle A B C to the Triangle B A D; and E A to A B ‡, as the Triangle E A D to the Triangle B A D; therefore, as the Triangle A B C is to the

Triangle B A D, * ſo ſhall the Triangle E A D be to * 11. 5. the Triangle B A D. Whence the Triangles A B C, A D E, have the ſame Proportion to the Triangle B A D; and ſo the Triangle A B C is ‡ equal to the ‡ 7. 5. Triangle A D E. Therefore, *equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and thoſe Triangles that have one Angle of the one equal to one Angle of the other, and have alſo the Sides about the equal Angles reciprocal, are equal*; which was to be demonſtrated.

PROPOSITION XVI.

THEOREM.

If four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional.

LET four Right Lines A B, C D, E, F, be proportional, ſo that A B be to C D, as E is to F. I ſay, the Rectangle contained under the Right Line A B and F, is equal to the Rectangle contained under the Right Lines C D and E.

For, draw A G and C H, from the Points A and C, at Right Angles to A B and C D; and make A G equal to F, and C H equal to E; and let the Parallelograms B G, D H, be compleated.

Then, becauſe A B is to C D, as E is to F; and ſince C H is equal to E, and A G to F; it ſhall be, as A B is to C D, ſo is C H to A G. Therefore, the Sides that are about the equal Angles of the Parallelograms B G, D H, are reciprocal; and ſince thoſe Parallelograms are equal *, that have the Sides about * 14 *of this.* the equal Angles reciprocal; therefore the Parallelogram B G is equal to the Parallelogram D H. But the Parallelogram B G is equal to that contained under A B and F; for A G is equal to F, and the Parallelogram D H equal to that contained under C D and E, ſince C H is equal to E. Therefore the Rectangle contained under A B and F, is equal to that contained under C D and E.

And if the Rectangle contained under A B and F be equal to the Rectangle contained under C D and E; I ſay, the four Right Lines are Proportionals, *viz.* as A B is to C D, ſo is E to F.

For, the ſame Conſtruction remaining, the Rectangle contained under A B and F is equal to that contained under C D and E; but the Rectangle contained under A B and F is the Rectangle B G; for A G is equal to F; and the Rectangle contained under C D and E is the Rectangle D H; for C H is equal to E. Therefore the Parallelogram B G ſhall be equal to the Parallelogram D H, and they are equiangular; but the Sides of equal and equiangular Parallelograms, which are about the equal Angles, are
* 14 of this. * reciprocal. Wherefore, as A B is to C D, ſo is C H to A G; but C H is equal to E, and A G to F; therefore, as A B is to C D, ſo is E to F. Wherefore, *if four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional*; which was to be demonſtrated.

PROPOSITION XVII.

THEOREM.

If three Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three Right Lines are proportional.

LET there be three Right Lines, A, B, C, proportional; and let A be to B, as B is to C. I ſay, the Rectangle, contained under A and C, is equal to the Square of B.

For, make D equal to B.

Then, becauſe A is to B as B is to C; and B is
* 7. 5. equal to D; it ſhall be *, as A is to B ſo is D to C. But, if four Right Lines be Proportionals, the Rectangle

gle contained under the Extremes is † equal to the † 16 of this. Rectangle under the Means. Therefore the Rectangle contained under A and C is equal to the Rectangle under B and D: But the Rectangle under B and D is equal to the Square of D; for B is equal to D: Wherefore the Rectangle contained under A, C, is equal to the Square of B.

And if the Rectangle contained under A and C be equal to the Square of B: I ſay, as A is to B, ſo is B to C.

For, the ſame Conſtruction remaining, the Rectangle contained under A and C, is equal to the Square of B; but the Square of B is the Rectangle contained under B and D; for B is equal to D; and the Rectangle contained under A and C ſhall be equal to the Rectangle contained under B and D. But if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, the four Right Lines ſhall be * Proportionals. Therefore A is * 16 of this. to B, as D is to C; but B is equal to D. Wherefore A is to B, as B is to C. Therefore, *if three Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three Right Lines are proportional*; which was to be demonſtrated.

PROPOSITION XVIII.

PROBLEM.

Upon a given Right Line, to deſcribe a Right-lined Figure, ſimilar, and ſimilarly ſituate, to a Right-lined Figure given.

LET A B be the Right Line given, and C E the Right-lined Figure. It is required to deſcribe upon the Right Line A B a Figure ſimilar, and ſimilarly ſituate, to the Right-lined Figure C E.

Join D F, and make, * at the Points A and B, with * 23. 1. the Line A B, the Angles G A B, A B G, ſeverally equal to the Angles C and C D F. Whence the other Angle C F D is † equal to the other Angle † Cor. 32. 1. A G B; and ſo the Triangle F C D is equiangular to the Triangle G A B: And, conſequently, as F D is

‡ 4 of this. to G B, ſo is ‡ F C to G A; and ſo is C D to A B. Again, make the Angles B G H, G B H, at the Points B and G, with the Right Line B G, ſeverally equal to the Angles E F D, E D F; then the remaining Angle † Cor. 32. 1. at E is † equal to the remaining Angle at H. Therefore the Triangle F D E is equiangular to the Triangle G B H; and, conſequently, as F D is to ‖ 4 of this. G B, ſo is ‖ F E to G H; and ſo E D to H B. But it has been proved, that F D is to G B, as F C is to G A, and as C D to A B. And therefore, as F C * 11. 5. is to A G, ſo is * C D to A B; and ſo F E to G H; and ſo E D to H B. And becauſe the Angle C F D is equal to the Angle A G B; and the Angle D F E equal to the Angle B G H; the whole Angle C F E ſhall be equal to the whole Angle A G H. By the ſame Reaſon, the Angle C D E is equal to the Angle A B H; and the Angle at C equal to the Angle at A; and the Angle E equal to the Angle H. Therefore the Figure A H is equiangular to the Figure C E; and they have the Sides about the equal Angles proportional. Conſequently, the Right-lined Figure * Def. 1. of this. A H will be * ſimilar to the Right-lined Figure C E. Therefore, *there is deſcribed upon the given Right Line* A B *the Right-lined Figure* A H, *ſimilar, and ſimilarly ſituate, to the given Right-lined Figure* C E; which was to be done.

PROPOSITION XIX.

THEOREM.

Similar Triangles are in the duplicate Proportion of their homologous Sides.

LET A B C, D E F, be ſimilar Triangles, having the Angle B equal to the Angle E; and let A B be to B C, as D E is to E F, ſo that B C be the Side homologous to E F. I ſay, the Triangle A B C, to the Triangle D E F, has a duplicate Proportion to that of the Side B C to the Side E F.

* 11 of this. For, take * B G a third Proportion to B C and E F; that is, let B C be to E F as E F is to B G, and join G A.

Then, becauſe A B is to B C, as D E is to E F; it ſhall be (by Alternation), as A B is to D E, ſo is B C to E F; but as B C is to E F, ſo is E F to B G. Therefore, ‡ 11. 5. as A B is to D E, ſo is ‡ E F to B G: Conſequently,

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quently, the Sides that are about the equal Angles of the Triangles A B G, D E F, are reciprocal: But thoſe Triangles that have one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles reciprocal, are ‡ equal. Therefore the Triangle A B G is equal to the Triangle D E F; and becauſe B C is to E F, as E F is to B G; and if three Right Lines be proportional, the firſt has * a duplicate Proportion to the third, of what it has to the ſecond; B C to B G ſhall have a duplicate Proportion of that which B C has to E F; and as B C is to B G, ſo is the Triangle A B C to the Triangle A B G; whence the Triangle A B C bears to the Triangle A B G a duplicate Proportion to what B C doth to E F; but the Triangle A B G is equal to the Triangle D E F. Therefore the Triangle A B C, to the Triangle D E F, ſhall be in the duplicate Proportion of that which the Side B C has to the Side E F. Wherefore, *ſimilar Triangles are in the duplicate Proportion of their homologous Sides*; which was to be demonſtrated.

‡ 15 *of this.*

* *Def.* 10. 5.

Coroll. From hence it is manifeſt, if three Right Lines be proportional; then, as the firſt is to the third, ſo is a Triangle made upon the firſt, to a ſimilar and ſimilarly deſcribed Triangle upon the ſecond; becauſe it has been proved, that as C B is to B G, ſo is the Triangle A B C to the Triangle A B G; that is, to the Triangle D E F; *which was to be demonſtrated.*

PROPOSITION XX.

Theorem.

Similar Polygons are divided into ſimilar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other.

LET A B C D E, F G H K L, be ſimilar Polygons, and let the Side A B be homologous to the Side F G. I ſay, the Polygons A B C D E, F G H K L, are divided into equal Numbers of ſimilar Triangles, and homologous to the Wholes; and the Polygon A B C D E, to the Polygon F G H K L, is in the duplicate Proportion of that which the Side AB has to the Side FG.

For let BE, EC, GL, LH, be joined.

Then, becauſe the Polygon ABCDE is ſimilar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL. Now, ſince ABE, FGL, are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles proportional; the Triangle ABE will be * equiangular to the Triangle FGL, and alſo ſimilar to it. Therefore the Angle ABE is equal to the Angle FGL; but the whole Angle ABC is * equal to the whole Angle FGH, becauſe of the Similarity of the Polygons; therefore the remaining Angle EBC is equal to the remaining Angle LGH: (And ſince by the Similarity of the Triangles ABE, FGL), as EB is to BA, ſo is LG to GF; and ſince, alſo, by the Similarity of the Polygons, AB is to BC as FG is to GH; it ſhall be †, by Equality of Proportion, as EB is to BC, ſo is LG to GH; that is, the Sides about the equal Angles EBC, LGH, are proportional. Wherefore the Triangle EBC is equiangular to the Triangle LGH, and, conſequently, alſo ſimilar to it. For the ſame Reaſon, the Triangle ECD is likewiſe ſimilar to the Triangle LHK; therefore the ſimilar Polygons ABCDE, FGHKL, are divided into equal Numbers of ſimilar Triangles.

* 6 of this.
* Def. 1. of this.
† 22. 5.

I ſay, they are alſo homologous to the Wholes; that is, that the Triangles are proportional, and the Antecedents are ABE, EBC, ECD; and their Conſequents FGL, LGH, LHK: And the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of an homologous Side of the one, to an homologous Side of the other; that is, as AB to FG.

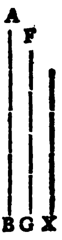
For, becauſe the Triangle ABE is ſimilar to the Triangle FGL, the Triangle ABE ſhall be * to the Triangle FGL, in the duplicate Proportion of BE to GL: For the ſame Reaſon, the Triangle BEC, to the Triangle GLH, is * in a duplicate Proportion of BE to GL: Therefore the Triangle ABE is † to the Triangle FGL, as the Triangle BEC is to the Triangle GLH. Again, becauſe the Triangle EBC,

* 19 of this.
† 11. 5.

is ſimilar to the Triangle L G H, the Triangle E B C to the Triangle L G H, ſhall be in the duplicate Proportion of the Right Line C E to the Right Line H L; and ſo, likewiſe, the Triangle E C D to the Triangle L H K, ſhall be in the duplicate Proportion of C E to H L. Therefore the Triangle B E C is to the Triangle L G H, as the Triangle C E D is to the Triangle L H K. But it has been proved, that the Triangle E B C is to the Triangle L G H, as the Triangle A B E is to the Triangle F G L. Therefore, as the Triangle A B E is to the Triangle F G L, ſo is the Triangle B E C to the Triangle G H L; and ſo is the Triangle E C D to the Triangle L H K. But as one of the Antecedents is to one of the Conſequents, ſo are ‡ all the Antecedents to all the Conſequents. ‡ 12. 5. Wherefore, as the Triangle A B E is to the Triangle F G L, ſo is the Polygon A B C D E to the Polygon F G H K L: But the Triangle A B E, to the Triangle F G L, * is in the duplicate Proportion of the * 19 of this. homologous Side A B to the homologous Side F G; for ſimilar Triangles are in the duplicate Proportion of the homologous Sides. Wherefore the Polygon A B C D E, to the Polygon F G H K L, is in the duplicate Proportion of the homologous Side A B to the homologous Side F G. Therefore, *ſimilar Polygons are divided into ſimilar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other*; which was to be demonſtrated.

It may be demonſtrated, after the ſame manner, that ſimilar quadrilateral Figures are to each other in the duplicate Proportion of their homologous Sides; and this has been already proved in Triangles.

Coroll. 1. Therefore, univerſally, ſimilarly Right-lined Figures are to one another in the duplicate Proportion of their homologous Sides; and if X be taken a third Proportional to A B and F G, then A B will have to X a duplicate Proportion of that which A B has to F G; and a Polygon to a Polygon, and a quadrilateral Figure to a quadrilateral Figure, will be in the duplicate Proportion of that which one homologous Side has to the other; that is, A B to F G; but this has been proved in Triangles.



2. Therefore, univerſally, it is manifeſt, if three Right Lines be proportional, as the firſt is to the third; ſo is a Figure deſcribed upon the firſt, to a ſimilar and ſimilarly deſcribed Figure on the ſecond; *which was to be demonſtrated.*

PROPOSITION XXI.

THEOREM.

Figures that are ſimilar to the ſame Right-lined Figure, are alſo ſimilar to one another.

LET each of the Right-lined Figures A, B, be ſimilar to the Right-lined Figure C. I ſay, the Right-lined Figure A is alſo ſimilar to the Right-lined Figure B.

* *Def.* 1. *of this.*

For, becauſe the Right-lined Figure A is ſimilar to the Right-lined Figure C, it ſhall be * equiangular thereto; and the Sides about the equal Angles proportional. Again, becauſe the Right-lined Figure B is ſimilar to the Right-lined Figure C, it ſhall * be equiangular thereto; and the Sides about the equal Angles will be proportional. Therefore each of the Right-lined Figures A, B, are equiangular to C, and they have the Sides about the equal Angles proportional. Wherefore the Right-lined Figure A is equiangular to the Right-lined Figure B; and the Sides about the equal Angles are proportional. Wherefore, A *is ſimilar to* B; which was to be demonſtrated.

PROPOSITION XXII.

THEOREM.

If four Right Lines be proportional, the Right-lined Figure, ſimilar and ſimilarly deſcribed upon them, ſhall be proportional; and if ſimilar Right-lined Figures ſimilarly deſcribed upon Lines be proportional, then the Right Lines ſhall be alſo proportional.

LET four Right Lines AB, CD, EF, GH, be proportional; and as AB is to CD, ſo let EF be to GH.

Now, let the ſimilar Figures K A B, L C D, be ſimilarly deſcribed ‖ upon A B, C D; and the ſimilar ‖ 18 of this. Figures M F, N H, ſimilarly deſcribed upon the Right Lines E F, G H. I ſay, as the Right-lined Figure K A B is to the Right-lined Figure L C D, ſo is the Right-lined Figure M F to the Right-lined Figure N H.

For, take * X a third Proportional to A B, C D; * 11 of this. and O a third Proportional to E F, G H.

Then, becauſe A B is to C D, as E F is to G H; and as C D is to X, ſo is G H to O; it ſhall be †, by † 22. 3. Equality of Proportion, as A B is to X, ſo is E F to O. But A B is to X, as the Right-lined Figure K A B is ‡ to the Right-lined Figure L C D; and as E F is to O, ſo is ‡ the Right-lined Figure M F, to ‡ Cor. 20. the Right-lined Figure N H. Therefore, as the Right- of this. lined Figure K A B is to the Right-lined Figure L C D, ſo is * the Right-lined Figure M F to the Right-lined * 11. 5. Figure N H.

And, if the Right-lined Figure K A B be to the Right-lined Figure L C D, as the Right-lined Figure M F is to the Right-lined Figure N H; I ſay, as A B is to C D, ſo is E F to G H.

For, make † E F to P R, as A B is to C D, and de- † 12 of this. ſcribe upon P R a Right-lined Figure S R ſimilar, and alike ſituate, to either of the Figures M F, N H.

Then, becauſe A B is to C D, as E F is to P R; and there are deſcribed upon A B, C D, ſimilar and alike ſituate Right-lined Figures K A B, L C D; and upon E F, P R, ſimilar and alike ſituate Figures M F, S R; it ſhall be (by what has been already proved), as the Right-lined Figure K A B is to the Right-lined Figure L C D, ſo is the Right-lined Figure M F to the Right-lined Figure S R: But (by the *Hyp.*) as the Right-lined Figure K A B is to the Right-lined Figure L C D, ſo is the Right-lined Figure M F to the Right-lined Figure N H. Therefore, as the Right-lined Figure M F is to the Right-lined Figure N H, ſo is the Right-lined Figure M F to the Right-lined Figure S R: And ſince the Right-lined Figure M F has the ſame Proportion to N H, as it hath to S R, the Right-lined Figure N H ſhall be ‡ equal to the ‡ 9. 5. Right-lined Figure S R; it is alſo ſimilar to it, and alike aſcribed; therefore G H is equal to P R. And becauſe A B is to C D, as E F is to P R; and P R is equal to G H; it ſhall be as A B is to C D, ſo is E F to G H. Therefore, *if four Right Lines be proportional, the Right-lined Figures, ſimilar and ſimilarly deſcribed upon them, ſhall be proportional; and if ſimilar Right-lined Figures, ſimilarly deſcribed upon Lines, be proportional, then the Right Lines ſhall alſo be proportional;* which was to be demonſtrated.

LEMMA.

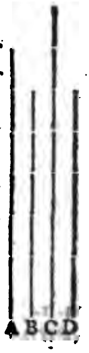
Any three Right Lines A, B, *and* C, *being given, the Ratio of the firſt* A, *to the third* C, *is equal to the Ratio compounded of the Ratio of the firſt* A *to the ſecond* B, *and of the Ratio of the ſecond* B *to the third* C.

FOR Example, Let the Number 3 *be the Exponent or Denominator of the Ratio of* A *to* B; *that is, let* A *be three Times* B, *and let the Number* 4 *be the Exponent of the Ratio of* B *to* C; *then the Number* 12, *produced by the Multiplication of* 4 *and* 3, *is the compounded Exponent of the Ratio of* A *to* C: *For, ſince* A *contains* B *thrice, and* B *contains* C *four Times,* A *will contain* C *thrice four Times, that is,* 12 *Times. This is alſo true of other Multiples, or Submultiples; but this Theorem may be univerſally demonſtrated thus: The Quantity of the Ratio of* A *to* B, *is the Number* $\frac{A}{B}$, *viz. which, multiplying the Conſequent, produced the Antecedent. So likewiſe the Quantity of the Ratio of* B *to* C, *is* $\frac{B}{C}$. *And theſe two Quantities, multiplied by each other, produce the Number* $\frac{A\times B}{B\times C}$ *which is the Quantity of the Ratio that the Rectangle, comprehended under the Right Lines* A *and* B, *has to the Rectangle comprehended under the Right Lines* B *and* C; *and ſo the ſaid Ratio of the Rectangle under* A *and* B, *to the Rectangle under* B *and* C, *is that which, in the Senſe of* Def. 5. *of this Book, is compounded of the Ratios of* A *to* B, *and* B *to* C; *but (by* 1. 6.) *the Rectangle contained under* A *and* B, *is to the Rectangle contained under* B *and* C, *as* A *is to* C; *therefore the Ratio of* A *to* C, *is equal to the*

the Ratio compounded of the Ratios of A *to* B, *and of* B *to* C.

If any four Right Lines A, B, C, *and* D, *be propoſed, the Ratio of the firſt* A *to the fourth* D, *is equal to the Ratio compounded of the Ratio of the firſt* A *to the ſecond* B, *and of the Ratio of the ſecond* B *to the third* C, *and of the Ratio of the third* C *to the fourth* D.

For, in three Right Lines A, C, *and* D, *the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratios of* A *to* C, *and of* C *to* D; *and it has been already demonſtrated, that the Ratio of* A *to* C *is equal to the Ratio compounded of the Ratios of* A *to* B, *and of* B *to* C: *Therefore the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratios of* A *to* B, *of* B *to* C, *and of* C *to* D. *After the ſame manner we demonſtrate, in any Number of Right Lines, that the Ratio of the firſt to the laſt is equal to the Ratio compounded of the Ratios of the firſt to the ſecond, of the ſecond to the third, of the third to the fourth, and ſo on to the laſt.*



This is true of any other Quantities beſides Right Lines, which will be manifeſt, if the ſame Number of Right Lines A, B, C, &c. *as there are Magnitudes, be aſſumed in the ſame Ratio,* viz. *ſo that the Right Line* A *is to the Right Line* B, *as the firſt Magnitude is to the ſecond, and the Right Line* B *to the Right Line* C, *as the ſecond Magnitude is to the third, and ſo on. It is manifeſt (by* 22. 5.), *by Equality of Proportion, that the firſt Right Line* A *is to the laſt Right Line, as the firſt Magnitude is to the laſt; but the Ratio of the Right Line* A *to the laſt Right Line is equal to the Ratio compounded of the Ratios of* A *to* B, B *to* C, *and ſo on to the laſt Right Line: But (by the* Hyp.) *the Ratio of any one of the Right Lines to that neareſt to it, is the ſame as the Ratio of a Magnitude of the ſame Order to that neareſt it. And therefore the Ratio of the firſt Magnitude to the laſt, is equal to the Ratio compounded of the Ratios of the firſt Magnitude to the ſecond, of the ſecond to the third, and ſo on to the laſt*; which was to be demonſtrated.

PROPOSITION XXIII.

Theorem.

Equiangular Parallelograms have the Proportion to one another that is compounded of their Sides.

LET AC, CE, be equiangular Parallelograms, having the Angle BCD equal to the Angle ECG. I ſay, the Parallelogram AC to the Parallelogram CF, is in the Proportion compounded of their Sides, *viz.* compounded of the Proportion of BC to CG, and of DC to CE.

For, let BC be placed in the ſame Right Line with CG.

* 14. 1. Then DC ſhall be * in a ſtrait Line with CE, and
† 12 *of this.* compleat the Parallelogram DG; and then †, as BC is to CG, ſo is ſome Right Line K to L; and as DC is to CE, ſo let L be to M.

Then the Proportions of K to L, and of L to M, are the ſame as the Proportions of the Sides, *viz.* of BC to CG, and DC to CE; but the Proportion
‡ *Lemma preced.* of K to M is ‡ compounded of the Proportion of K to L, and of the Proportion of L to M. Therefore, alſo, K to M hath a Proportion compounded of the Sides. Then, becauſe BC is to CG as the Parallelogram AC is * to the Parallelogram CH: And ſince
* 1 *of this.*
† 11. 5. BC is to CG, as K is to L; it ſhall be †, as K is to L, ſo is the Parallelogram AC to the Parallelogram CH. Again, becauſe DC is to CE, as the Parallelogram CH is to the Parallelogram CF; and ſince as DC is to CE, ſo is L to M; therefore as L is to M, ſo ſhall † the Parallelogram CH be to the Parallelogram CF: And conſequently ſince it has been proved that K is to L, as the Parallelogram AC is to the Parallelogram CH; and as L is to M, ſo is the Parallelogram CH to the Parallelogram CF; it ſhall
‡ 22. 5. be ‡, by Equality of Proportion, as K is to M, ſo is the Parallelogram AC to the Parallelogram CF; but K to M hath a Proportion compounded of the Sides: Therefore, alſo, the Parallelogram AC, to the Parallelogram CF, hath a Proportion compounded of the Sides. Wherefore, *equiangular Parallelograms have the*

the Proportion to one another that is compounded of their Sides; which was to be demonſtrated.

PROPOSITION XXIV.

Theorem.

In every Parallelogram, the Parallelograms that are about the Diameter, are ſimilar to the Whole, and alſo to one another.

LET A B C D be a Parallelogram, whoſe Diameter is A C; and E G, H K, be Parallelograms about the Diameter A C. I ſay, the Parallelograms E G, H K, are ſimilar to the Whole A B C D, and alſo to each other.

For, becauſe E F is drawn parallel to B C, the Side of the Triangle A B C, it ſhall be *, as B E is to E A, * 2 *of this.* ſo is C F to F A. Again, becauſe F G is drawn parallel to C D, the Side of the Triangle A C D, it ſhall be as C F is to F A, ſo is * D G to G A. But C F is to F A (as has been proved), as B E is to E A. Therefore, as B E is to E A, ſo is † D G to G A; † 11. 5. and by compounding, as B A is to A E, ſo is ‡ D A ‡ 18. 5. to A G; and, by Alternation, as B A is to A D, ſo is A E to A G. Therefore the Sides of the Parallelograms A B C D, E G, which are about the common Angle B A D are proportional. And becauſe G F is parallel to D C, the Angle A G F is * equal to the * 29. 1. Angle A D C, and the Angle G F A equal to the Angle D C A; and the Angle D A C is common to the Triangles A D C, A G F. Wherefore the Triangle A D C will be equiangular to the Triangle A G F. For the ſame Reaſon, the Triangle A C B is equiangular to the Triangle A F E. Therefore the whole Parallelogram A B C D is equiangular to the Parallelogram E G; and ſo A D is to D C, as A G is † to † 4 *of this.* G F; D C is to C A, as G F is to F A; and A C is to C B, as A F is to F E; and, moreover, C B is to B A, as F E is to E A. Wherefore, ſince it has been proved, that D C is to C A, as G F is to F A; and A C is to C B, as A F is to F E; it ſhall be, by Equality of Proportion, as D C is to C B, ſo is G F to F E. Therefore the Sides that are about the equal

Angles of the Parallelograms A B C D, E G, are proportional; and, accordingly, the Parallelogram A B C D is ſimilar to the Parallelogram E G. For the ſame Reaſon, the Parallelogram A B C D is ſimilar to the Parallelogram K H. Therefore both the Parallelograms E G, H K, are ſimilar to the Parallelogram A B C D. But Right-lined Figures that † 21 *of this.* are ſimilar to the † ſame Right-lined Figure, are ſimilar to one another. Therefore the Parallelogram E G is ſimilar to the Parallelogram H K. And ſo, *in every Parallelogram, the Parallelograms that are about the Diameter, are ſimilar to the Whole, and alſo to one another*; which was to be demonſtrated.

PROPOSITION XXV.

Problem.

To deſcribe a Right-lined Figure ſimilar to a Right-lined Figure which ſhall be given, and equal to another Right-lined Figure given.

LET A B C, and D, be two given Right-lined Figures; it is required to deſcribe another Figure, ſimilar to A B C and equal to D.

* 44. 1. On the Side B C of the given Figure A B C *, make the Parallelogram B E equal to the Right-lined Figure A B C; and on the Side C E make * the Parallelogram C M equal to the Right-lined Figure D, in the Angle F C E, equal to the Angle C B L. † 14. 1. Then B C, C F, as alſo L E, E M, will be † in two ‡ 13 *of this.* ſtrait Lines. Find ‡ G H a mean Proportional between B C and C F; and on G H let there be de* 18 *of this.* ſcribed * the Right-lined Figure K G H, ſimilar, and alike ſituate, to the Right-lined Figure A B C.

And then, becauſe B C is to G H, as G H is to C F; and ſince, when three Right Lines are proportional, the firſt is to the third, as the Figure deſcribed † *Cor.* 20. *of this.* on the firſt is † to a ſimilar and alike ſituate Figure deſcribed on the ſecond; it ſhall be, as B C is to C F, ſo is the Right-lined Figure A B C to the Right-lined ‡ 1 *of this.* Figure K G H. But as B C is to C F, ſo is ‡ the Parallelogram B E to the Parallelogram E F. Therefore, as the Right-lined Figure A B C is to the Right-lined Figure K G H, ſo is the Parallelogram B E to the Parallelogram E F. Wherefore (by Alternation), as the Right-lined Figure A B C is to the Parallelogram B E, ſo is the Right-lined Figure K G H to the Parallelogram E F. But the Right-lined Figure A B C is equal to the Parallelogram B E. Therefore the Right-lined Figure K G H is alſo equal to the Parallelogram E F. But the Parallelogram E F is equal to the Right lined Figure D. Therefore the Right-lined Figure K G H is equal to D. But K G H is ſimilar to A B C. Conſequently, *there is deſcribed the Right-lined Figure* K G H *ſimilar to the given Figure* A B C, *and equal to the given Figure* D; which was to be done.

PROPOSITION XXVI.

THEOREM.

If from a Parallelogram be taken away another ſimilar to the Whole, and in like manner ſituate, having alſo an Angle common with it; then is that Parallelogram about the ſame Diameter with the Whole.

LET the Parallelogram A F be taken away from the Parallelogram A B C D, ſimilar to A B C D, and in like manner ſituate, having the Angle D A B common. I ſay, the Parallelogram A B C D is about the ſame Diameter with the Parallelogram A F.

For, if it be not, let A H C be the Diameter of the Parallelogram B D, and let G F be produced to H; alſo let H K be drawn parallel to A D, or B C.

Then, becauſe the Parallelogram A B C D is about the ſame Diameter as the Parallelogram K G, the Parallelogram A B C D ſhall be * ſimilar to the * *24 of this.*
Parallelogram K G; and ſo, as D A is to A B, ſo is † G A to A K. But becauſe of the Similarity of the † *Def.* 1. *of*
Parallelograms * A B C D, E G; as D A is to A B, *this.* * *Hyp.*
ſo is G A to A E. And therefore, as G A is ‡ to ‡ 11. 5.
A E, ſo is G A to A K. And ſince G A has the ſame Proportion to A K as to A E, A E is † equal to † 9. 5.
A K, the leſs to a greater, or the greater to a leſs; which is abſurd. Therefore the Parallelogram A B C D is not about the ſame Diameter as the Parallelogram A H. And therefore it will be about the

ſame Diameter with the Parallelogram A F. Therefore, *if from a Parallelogram be taken away another ſimilar to the Whole, and in like manner ſituate, having alſo an Angle common with it; then is that Parallelogram about the ſame Diameter with the Whole*; which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

Of all Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar, and alike ſituate, to that deſcribed on the half Line, the greateſt is that which is applied to the half Line, and it is ſimilar to the Defect.

LET A B be a Right Line, biſected in the Point C; and let the Parallelogram A D be applied to the Right Line A B, wanting in Figure the Parallelogram C E, ſimilar and alike ſituate to that deſcribed on half of the Right Line A B. I ſay, A D is the greateſt of all Parallelograms applied to the Right Line A B, wanting in Figure by Parallelograms ſimilar and alike ſituate to C E. For, let the Parallelogram A F be applied to the Right Line A B, wanting in Figure the Parallelogram H K, ſimilar and alike ſituate to the Parallelogram C E. I ſay, the Parallelogram A D is greater than the Parallelogram A F.

For, becauſe the Parallelogram C E is ſimilar to the Parallelogram H K, they ſtand * about the ſame Diameter. Let D B, their Diameter, be drawn, and the Figure deſcribed; then, ſince the Parallelogram C F is † equal to F E, let H K, which is common, be added; and the Whole C H is equal to the Whole K E. But C H is ‡ equal to C G, becauſe the Right Line A C is equal to C B; therefore C G is equal to K E; add the common Parallelogram C F, and the Whole A F is equal to the Gnomon L N M; and ſo C E, that is the Parallelogram A D, is greater than the Parallelogram A F. Therefore, *of all the Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar, and alike ſituate, to that deſcribed on the half Line, the greateſt is that which is applied to the half Line, and it is ſimilar to the Defect*; which was to be demonſtrated.

* 26 *of this.*
† 43. 1.
‡ 36. 1.

PROPOSITION XXVIII.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right-lined Figure given, deficient by a Parallelogram, which is ſimilar to another given Parallelogram; but it is neceſſary that the Right-lined Figure given, to which the Parallelogram to be applied muſt be equal, be not greater than the Parallelogram which is applied to the half Line, ſince the Defects muſt be ſimilar, viz. *the Defect of the Parallelogram applied to the half Line, and the Defect of the Parallelogram to be applied.*

LET A B be a given Right Line, and let the given Right-lined Figure, to which the Parallelogram to be applied to the Right Line A B muſt be equal, be C, which muſt not be greater than the Parallelogram applied to the half Line, the Defects being ſimilar; and let D be the Parallelogram, to which the Defect of the Parallelogram to be applied is ſimilar. Now it is required to apply a Parallelogram equal to the given Right lined Figure C to the given Right Line A B, deficient by a Parallelogram ſimilar to D.

Let A B be biſected in E, and on E B deſcribe* the Parallelogram E B F G, ſimilar and alike ſituate to D; and compleat the Parallelogram A G. * 18 *of this.*

Now, A G is either equal to C, or greater than it, becauſe of the Determination. If A G be equal to C, what was propoſed will be done; for the Parallelogram A G is applied to the Right Line A B equal to the given Right-lined Figure C, deficient by the Parallelogram E F, ſimilar to the Parallelogram D. But, if it be not equal, then H E is greater than C; but E F is equal to H E; therefore E F ſhall alſo be greater than C. Now make † the Parallelogram K L M N ſimilar and alike ſituate to D, and equal to the Exceſs, by which E F exceeds C. But D is ſimilar to E F; wherefore K M ſhall alſo be ſimilar to E F. Therefore let the Right Line K L be homologous to G E, and L M to G F: Then, becauſe E F † 25 *of this.*

is equal to C and K M together, E F will be greater than K M; and ſo the Right Line G E is greater than K L, and G F than L M. Make G X equal to K L, and G O equal to L M, and compleat the Parallelogram X G O P. Therefore X O is equal and ſimilar to K M; but K M is ſimilar to E F; therefore
* 21 of this. X O is * ſimilar to E F; and ſo X O is † about the
† 26 of this. ſame Diameter with F E: Let G P B be their Diameter, and the Figure be deſcribed.

Then, ſince E F is equal to C and K M together, and X O is equal to K M, the Gnomon Υ Φ Ψ remaining is equal to the remaining Figure C; and becauſe O R is equal to X S, let S R, which is common, be added; then the Whole O B is equal to the Whole X B; but X B is equal to T E, ſince the Side A E is equal to the Side E B. Wherefore T E is equal to O B. Add X S, which is common, and then the Whole T S is equal to the Whole Gnomon Υ Φ Ψ; but the Gnomon Υ Φ Ψ has been proved equal to C; and T S ſhall be equal to C; and ſo, *the Parallelogram* T S *is applied to the Right Line* A B, *equal to the given Right lined Figure* C, *and deficient by a Parallelogram* S R, *ſimilar to the Parallelogram* D, *becauſe* S R *is ſimilar to* F E; which was to be done.

PROPOSITION XXIX.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right-lined Figure given, exceeding by a Parallelogram, which ſhall be ſimilar to another given Parallelogram.

LET A B be a given Right Line, and let C be the given Right-lined Figure, to which that to be applied to A B muſt be equal: Likewiſe, let D be the Parallelogram, to which the exceeding Parallelogram is to be ſimilar; it is required to apply a Parallelogram to the Right Line A B, equal to the given Right-lined Figure C, exceeding by a Parallelogram ſimilar to D.

Biſect A B in E, and let the Parallelogram E L be
* 18 of this. deſcribed * upon the Right Line E B, ſimilar and alike
† 25 of this. ſituate to D; and let † the Parallelogram G H be equal to E L and C together, but ſimilar to D, and alike ſituate;

ſituate; therefore G H is ſimilar to E L. Let K H be a Side homologous to F L, and K G to F E; then becauſe the Parallelogram G H is greater than the Parallelogram E L, the Right Line K H will be greater than F L, and K G greater than F E. Let F L, F E, be produced, and let F L M be equal to K H, F E N equal to K G, and compleat the Parallelogram M N; therefore M N is equal and ſimilar to G H; but G H is ſimilar to E L, and ſo M N ſhall be ‡ ſimilar to ‡ 21 *of this.*
E L; and, accordingly, E L is * about the ſame Di- * 26 *of this.*
ameter with M N. Let F X be their Diameter, and deſcribe the Figure.

Then, ſince G H is equal to E L and C together, as likewiſe to M N; therefore M N ſhall be equal to E L and C together. Let E L, which is common, be taken away; then the Gnomon Φ Υ Ψ remaining is equal to C; and ſince A E is equal to E B, the Parallelogram A N will be alſo equal to the Parallelogram E P, that is, to L O; and if B X, which is common, be added, then the whole Parallelogram A X is equal to the Gnomon Φ Υ Ψ; but the Gnomon Φ Υ Ψ is equal to C; therefore A X ſhall alſo be equal to C. Wherefore *the Parallelogram* A X *is applied to the given Right Line* A B, *equal to the given Right-lined Figure* C, *and exceeding by the Parallelogram* P O, *ſimilar to the Parallelogram* D; which was to be done.

PROPOSITION XXX.

Problem.

To cut a given terminate Right Line according to extreme and mean Ratio.

LET A B be a given terminate Line; it is required to cut the ſame according to extreme and mean Ratio.

Deſcribe * C B, the Square of A B; and apply the * 46. 1.
Parallelogram C D to A C, equal to the Square B C, exceeding † by the Figure A D ſimilar to B C; but † 29 *of this.*
B C is a Square; therefore A D ſhall alſo be a Square.

Now, becauſe B C is equal to C D, take away C E, which is common; then B F remaining ſhall be equal

to AD remaining; but BF is equiangular to AD;
therefore the Sides that are about the equal Angles are
‡ 14 of this. ‡ reciprocally proportional; and ſo as FE is to ED,
* 34. 1. ſo is AE to EB; but FE is *equal to AC, that is,
to AB; and ED to AE: Wherefore, as AB is to
AE, ſo is AE to EB; but AB is greater than AE;
† 14. 5. therefore AE is † greater than EB; and ſo the Right
Line AB is cut, according to extreme and mean Ratio, in the Point E; and AE is the greater Segment
thereof; *which was to be done.*

Otherwiſe thus: Let AB be the Right Line given; it is required to cut the ſame into extreme and mean Ratio.

‡ 11. 2. Divide ‡ AB ſo in C, that the Rectangle contained under AB, BC, be equal to the Square of AC.

Then, becauſe the Rectangle under AB, BC, is
* 17 of this. equal to the Square of AC, it ſhall be *, as AB is to
AC, ſo is AC to CB; and ſo, *the Right Line AB is cut into mean and extreme Ratio*; which was to be done.

PROPOSITION XXXI.

THEOREM.

Any Figure deſcribed upon the Side of a Right-angled Triangle, ſubtending the Right Angle, is equal to the two Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure.

LET ABC be a rectangular Triangle, having the Right Angle BAC. I ſay, the Figure deſcribed on BC is equal to the two Figures deſcribed on BA, AC, together, which are ſimilar and alike ſituate to the Figure deſcribed on BC.

For, draw the Perpendicular AD.

Then, becauſe the Right Line AD is drawn in the Right-angled Triangle ACB, from the Right Angle A, perpendicular to the Baſe BC; the Triangles ABD, ADC, which are about the Perpendicular
* 8 of this. AD, will be * ſimilar to the whole Triangle ABC,
and alſo to each other. Then, becauſe the Triangle ABC is ſimilar to the Triangle ABD, it ſhall be *,

as C B is to B A, ſo is B A to B D: And ſince, when three Right Lines are proportional, the firſt ſhall be † † Cor. 20. of this. to the third, as a Figure deſcribed on the firſt, to a ſimilar and alike ſituate Figure deſcribed on the ſecond; therefore, as C B is to B D, ſo is a Figure deſcribed on C B, to a ſimilar and alike ſituate Figure deſcribed on B A. For the ſame Reaſon, as B C is to C D, ſo is a Figure deſcribed on B C to one deſcribed on C A. Wherefore, alſo, as B C is to B D and D C together, ſo is ‡ the Figure deſcribed on B C to thoſe two toge- ‡ 24. 5. ther, that are deſcribed ſimilar and alike ſituate on B A and A C; but B C is equal to B D and D C together: Therefore the Figure deſcribed on B C is equal to thoſe together, which are deſcribed on B A and A C, ſimilar and alike ſituate to that on B C. Wherefore, *any Figure deſcribed upon the Side of a Right-angled Triangle, ſubtending the Right Angle, is equal to the Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure*; which was to be demonſtrated.

PROPOSITION XXXII.

Theorem.

If two Triangles, having two Sides proportional to two Sides, be ſo compounded, or ſet together, at one Angle, that their homologous Sides be parallel; then the other Sides of theſe Triangles will be in one ſtrait Line.

LET there be two Triangles A B C, D C E, having two Sides B A, A C, of the one, proportional to two Sides C D, D E, of the other, *viz.* let B A be to A C, as C D is to D E; alſo let A B be parallel to D C, and A C to D E. I ſay, B C, C E, are both in one ſtrait Line.

For, becauſe A B is parallel to D C, and the Right Line A C falls on them, the alternate Angles B A C, A C D, will be * equal to each other. And by the * 29. 1. ſame Reaſon, the Angle C D E is equal to the Angle A C D; wherefore the Angle B A C is equal to the Angle C D E. Then, becauſe A B C, D C E, are two Triangles, having one Angle A equal to one Angle

D, and the Sides about the equal Angles proportional, *viz.* BA to AC, as CD to DE; the Triangle ABC will be * equiangular to the Triangle DCE; wherefore the Angle ABC is equal to the Angle DCE; but the Angle ACD has been proved to be equal to the Angle BAC; therefore the whole Angle ACE is equal to the two Angles ABC, BAC; and if ACB, which is common, be added, then the Angles ACE, ACB, are equal to the Angles BAC, ACB, CBA; but the Angles BAC, ACB, CBA ‡, are equal to two Right Angles. Therefore the Angles ACE, ACB, will alſo be equal to two Right Angles; and ſo at the Point C, in the Right Line AC, two Right Lines BC, CE, tending contrary Ways, make the adjacent Angles ACE, ACB, equal to two Right Angles; therefore BC ſhall be † in the ſame Right Line with CE. Wherefore, *if two Triangles having two Sides proportional to two Sides, be ſo compounded or ſet together, at one Angle, that their homologous Sides be parallel; then the other Sides of theſe Triangles will be in one ſtrait Line*; which was to be demonſtrated.

* 6 of this.
‡ 32. 1.
† 14. 1.

PROPOSITION XXXIII.

THEOREM.

In equal Circles the Angles have the ſame Proportion with the Circumferences on which they ſtand, whether the Angles be at the Centres, or at the Circumferences; and ſo likewiſe are the Sectors, as being at the Centres.

LET ABC, DEF, be equal Circles; and let the Angles BGC, EHF, be at their Centres G, H; and the Angles BAC, EDF, at their Circumferences. I ſay, as the Circumference BC is to the Circumference EF, ſo is the Angle BGC to the Angle EHF; and ſo is the Angle BAC to the Angle EDF; and ſo is the Sector BGC to the Sector EHF.

For, aſſume any Number of continuous Circumferences CK, KL, each equal to BC; and alſo any Number FM, MN, each equal to EF; and join GK, GL, HM, HN.

Then, becauſe the Circumferences B C, C K, K L, are equal to each other; the Angles B G C, C G K, K G L, will be * alſo equal to one another; and ſo * 27. 3. the Circumference B L is the ſame Multiple of the Circumference B C, as the Angle B G L is of the Angle B G C. For the ſame Reaſon, the Circumference N E is the ſame Multiple of the Circumference E F, as the Angle E H N is of the Angle E H F; but if the Circumference B L be equal to the Circumference E N, then the Angle B G L ſhall be equal to the Angle E H N; and if the Circumference B L be greater than the Circumference E N, the Angle B G L will be greater than the Angle E H N; and if leſs, leſs. Therefore here are four Magnitudes, *viz.* the two Circumferences B C, E F, and the two Angles B G C, E H F; and ſince there are taken Equimultiples of the Circumference B C, and the Angle B G C, to wit, the Circumference B L, and the Angle B G L; as alſo Equimultiples of the Circumference E F, and the Angle E H F, *viz.* the Circumference E N, and the Angle E H N; and becauſe it is proved, if the Circumference B L exceeds the Circumference E N, the Angle B G L will likewiſe exceed the Angle E H N; and, if equal, equal; if leſs, leſs; it ſhall be, as the Circumference B C is to the Circumference E F, ſo is † the Angle B G C to the Angle E H F; † *Def.* 5. 5. but as the Angle B G C is to the Angle E H F, ſo is ‡ the Angle B A C to the Angle E D F, for the for- ‡ 15. 5. mer are * double to the latter: Therefore, as the Cir- * 20. 3. cumference B C is to the Circumference E F, ſo is the Angle B G C to the Angle E H F; and ſo the Angle B A C to the Angle E D F.

Wherefore, *in equal Circles, the Angles have the ſame Proportion as the Circumferences they ſtand on, whether they be at the Centres, or at the Circumferences.*

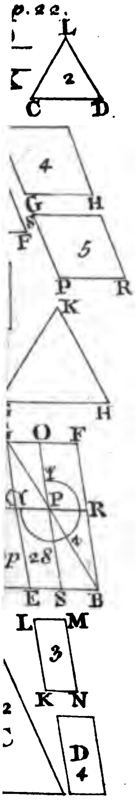
I ſay, moreover, that as the Circumference B C is to the Circumference E F, ſo is the Sector G B C to the Sector H F E.

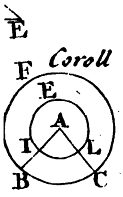
For, join B C, C K, and aſſume the Points X, O, in the Circumferences B C, C K; and join B X, X C, C O, O K.

Then, becauſe the two Sides B G, G C, are equal to the two Sides C G, G K, and they contain equal

† 4. 1. Angles, the Baſe BC ſhall be † equal to the Baſe CK; as likewiſe the Triangle GBC to the Triangle GCK. And, becauſe the Circumference BC is equal to the Circumference CK, and the Circumference remaining, which makes up the whole Circle ABC, is equal to the remaining Circumference, which makes up the ſame Circle ABC, the Angle BXC is equal to the Angle COK; and ſo the Segment BXC is ſimilar to the Segment COK; and they are upon equal Right Lines BC, CK; but ſimilar Segments of Circles, that ſtand upon equal Right * 24. 3. Lines, are * equal to each other: Therefore the Segment BXC is equal to the Segment COK. But the Triangle BGC is alſo equal to the Triangle CGK, and ſo the whole Sector BGC will be equal to the whole Sector CGK. By the ſame Reaſon, the Sector GKL will be equal to the Sector GBC, or GCK; therefore the three Sectors BGC, CGK, KGL, are equal to one another; ſo likewiſe are the Sectors HEF, HFM, HMN. Wherefore the Circumference LB is the ſame Multiple of the Circumference BC, as the Sector GBL is of the Sector GBC. For the ſame Reaſon, the Circumference NE is the ſame Multiple of the Circumference EF, as the Sector HEN is of the Sector HEF; but if the Circumference BL be equal to the Circumference EN, then the Sector BGL will be equal to the Sector EHN; and if the Circumference BL exceeds the Circumference EN, then the Sector BGL will alſo exceed the Sector EHN; and, if leſs, leſs. Therefore, ſince there are four Magnitudes, to wit, the two Circumferences BC, EF, and the two Sectors GBC, EHF; and there are taken the Circumference BL, and the Sector GBL, Equimultiples of the Circumference CB, and the Sector GBC; as alſo the Circumference EN, and the Sector HEN Equimultiples of the Circumference EF, and the Sector HEF; and becauſe it is proved, that, if the Circumference BL exceeds the Circumference EN, the Sector BGL will alſo exceed the Sector EHN; and, if † *Def.* 5. 5. equal, equal; if leſs, leſs; therefore, † *as the Circumference* BC *is to the Circumference* EF, *ſo is the Sector* GBC *to the Sector* HEF; which was to be demonſtrated.

Coroll.





OF MICH.

Coroll. 1. An Angle at the Centre of a Circle is to four Right Angles, as an Arc on which it ſtands is to the whole Circumference; for as the Angle B A C is to a Right Angle, ſo is the Arc B C to a Quadrant of the Circle: Wherefore, if the Conſequents be quadrupled, the Angle B A C ſhall be to four Right Angles as the Arc B C is to the whole Circumference.

2. The Arcs I L, B C, of unequal Circles, which ſubtend equal Angles, whether at their Centres, or Circumferences, are ſimilar; for I L, is to the whole Circumference I L E, as the Angle I A L is to four Right Angles; but as I A L, or B A C, is to four Right Angles, ſo is the Arc B C to the whole Circumference B C F. Therefore, as I L is to the whole Circumference I L E, ſo is B C to the whole Circumference B C F; and ſo the Arcs I L, B C, are ſimilar.

3. Two Semidiameters A B, A C, cut off ſimilar Arcs I L, B C, from concentric Circumferences.

The END *of the* SIXTH BOOK.

EUCLID's

ELEMENTS.

BOOK XI.

DEFINITIONS.

I. *A Solid is that which has Length, Breadth, and Thickneſs.*

II. *The Term of a Solid is a Superficies.*

III. *A Right Line is perpendicular to a Plane, when it makes Right Angles with all the Lines that touch it, and are drawn in the ſaid Plane.*

IV. *A Plane is perpendicular to a Plane, when all the Right Lines in one Plane, drawn at Right Angles to the common Section of the two Planes, are at Right Angles to the other Plane.*

V. *The Inclination of a Right Line to a Plane, is the acute Angle contained under that Line, and another Right one drawn in the Plane from that End of the inclining Line which is in the Plane, to the Point where a Right Line falls from the other End of the inclining Line perpendicular to the Plane.*

VI. *The Inclination of a Plane to a Plane, is the acute Angle contained under the Right Lines drawn in both the Planes to the ſame Point of their*

their common Interſection, and making Right Angles with it.

VII. *Planes are ſaid to be inclined ſimilarly, when the ſaid Angles of Inclination are equal.*

VIII. *Parallel Planes are ſuch, which, being produced, never meet.*

IX. *Similar ſolid Figures are ſuch, as are contained under equal Numbers of ſimilar Planes.*

X. *Equal and ſimilar ſolid Figures are thoſe, that are contained under equal Numbers of ſimilar and equal Planes.*

XI. *A ſolid Angle is the Inclination of more than two Right Lines, that touch one another, and are not in the ſame Superficies: Or, a ſolid Angle is that which is contained under more than two plane Angles, which are not in the ſame Superficies, but being all at one Point.*

XII. *A Pyramid is a ſolid Figure comprehended under divers Planes ſet upon one Plane, and put together at one Point.*

XIII. *A Priſm is a ſolid Figure contained under Planes, whereof the two oppoſite are equal, ſimilar, and parallel, and the other Parallelograms.*

XIV. *A Sphere is a ſolid Figure, made when the Diameter of a Semicircle remaining at Reſt, the Semicircle is turned about till it returns to the ſame Place from whence it begun to move.*

XV. *The Axis of a Sphere is that fixed Line, about which the Semicircle is turned.*

XVI. *The Centre of a Sphere is the ſame with that of the Semicircle.*

XVII. *The Diameter of a Sphere is a Right Line drawn through the Centre, and terminated on either Side by the Superficies of the Sphere.*

XVIII. *A Cone is a Figure deſcribed when one of the Sides of a Right-angled Triangle, containing the Right Angle, remaining fixed, the Triangle is turned about until it returns to the Place from whence*

whence it firſt began to move. And if the fixed Right Lines be equal to each other, that contains the Right Angle, then the Cone is a rectangular Cone; but, if it be leſs, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. *The Axis of a Cone is that fixed Right Line, about which the Triangle is moved.*

XX. *The Baſe of a Cone is the Circle deſcribed by the Right Line moved about.*

XXI. *A Cylinder is a Figure deſcribed by the Motion of a Right-angled Parallelogram, one of the Sides containing the Right Angle, remaining fixed while the Parallelogram is turned about to the ſame Place from whence it began to be moved.*

XXII. *The Axis of a Cylinder is that fixed Right Line, about which the Parallelogram is turned.*

XXIII. *And the Baſes of a Cylinder are the Circles that are deſcribed by the Motion of the two oppoſite Sides of the Parallelogram.*

XXIV. *Similar Cones and Cylinders are ſuch, whoſe Axes and Diameters of their Baſes are proportional.*

XXV. *A Cube is a ſolid Figure contained under ſix equal Squares.*

XXVI. *A Tetrahedron is a ſolid Figure contained under four equal equilateral Triangles.*

XXVII. *An Octahedron is a ſolid Figure contained under eight equal equilateral Triangles.*

XXVIII. *A Dodecahedron is a ſolid Figure contained under twelve equal equilateral and equiangular Pentagons.*

XXIX. *An Icoſahedron is a ſolid Figure contained under twenty equal equilateral Triangles.*

XXX. *A Parallelopipedon is a Figure contained under ſix quadrilateral Figures, whereof thoſe which are oppoſite are parallel.*

PROPOSITION I.

THEOREM.

One Part of a Right Line cannot be in a plane Superficies, and another Part above it.

FOR, if poſſible, let the Part A B of the Right Line A B C be in a plane Superficies, and the Part B C above the ſame.

There will be ſome Right Line in the aforeſaid Plane, which, with A B, will be but one ſtrait Line. Let this Line be D B.

Then the two given Right Lines A B C, A B D, have one common Segment A B, which is impoſſible; for one Right Line will not meet another in more Points than one. Wherefore, *one Part of a Right Line cannot be in a plane Superficies, and another Part above it*; which was to be demonſtrated.

PROPOSITION II.

THEOREM.

If two Right Lines cut each other, they are both in one Plane; and every Triangle is in one Plane.

LET two Right Lines A B, C D, cut each other in the Point E. I ſay, they are both in one Plane; and every Triangle is one Plane.

For, take any Points F and G, in the Right Lines A B, C D; and join C B, F G; and let there be drawn F H, G K. In the firſt Place, I ſay, the Triangle E B C is in one Plane.

For, if one Part F H C, or G B K, of the Triangle E B C, be in one Plane, and the other Part in another Plane; then one Part of each of the Lines E C, E B, ſhall be in one Plane, and the other Part in another Plane; which we have proved * to be abſurd. Therefore the Triangle E B C is one Plane; but both the Right Lines E C, E B, are in the ſame Plane as the

* 1 *of this.*

Triangle B C E is; and A B, C D, are both in the ſame Plane as E C, E B, are. Wherefore, *the Right Lines* A B, C D, *are both in one Plane*; *and every Triangle is in one Plane*; which was to be demonſtrated.

PROPOSITION III.

THEOREM.

If two Planes cut each other, their common Section will be a Right Line.

LET two Planes A B, B C, cut each other, whoſe common Section is the Line D B; I ſay, D B is a Right Line.

For if it be not, draw the Right Line D E B in the Plane A B, from the Point D to the Point B, and the Right Line D F B in the Plane B C.

Then two Right Lines D E B, D F B, have the ſame Terms, and include a Space, which is * abſurd. Therefore D E B, D F B, are not Right Lines. In the ſame manner we demonſtrate, that no other Line drawn from the Point D to the Point B, is a Right Line, beſides D B, the common Section of the Planes A B, B C. *If*, therefore, *two Planes cut each other, their common Section will be a Right Line*; which was to be demonſtrated.

* *Ax.* 10.

PROPOSITION IV.

THEOREM.

If to two Right Lines, cutting one another, a third ſtands at Right Angles in the common Section, it ſhall be alſo at Right Angles to the Plane drawn thro' the ſaid Lines.

LET the Right Line E F ſtand at Right Angles to the two Right Lines A B, C D, in the common Section E. I ſay, E F is alſo at Right Angles to the Plane drawn through A B, C D.

For, take the equal Right Lines E A, E B, C E,
D E; and thro' E any how draw the Right Line
G E H, and join A D, C B; and from the Point F, let
there be drawn F A, F G, F D, F C, F H, F B:
Then, becauſe two Right Lines A E, E D, are equal
to two Right Lines C E, E B, and they contain * the * 15. 1.
equal Angles A E D, C E B; the Baſe A D ſhall be †† 4. 1.
equal to the Baſe C B, and the Triangle A E D equal
to the Triangle C E B; and ſo, likewiſe, is the Angle
D A E equal to the Angle E B C; but the Angle
A E G is * equal to the Angle B E H; therefore
A G E, B E H, are two Triangles, having two Angles
of the one equal to two Angles of the other, each to
each, and one Side A E equal to one Side E B, *viz.*
thoſe that are at the equal Angles; and ſo the other
Sides of the one will be ‡ equal to the other Sides of ‡ 26. 1.
the other. Therefore G E is equal to E H, and A G
to B H; and ſince A E is equal to E B, and F E is
common and at Right Angles, the Baſe A F ſhall be
† equal to the Baſe F B: For the ſame Reaſon, like- † 4. 1.
wiſe, ſhall C F be equal to F D. Again, becauſe A D
is equal to C B, and A F to F B, the two Sides F A,
A D, will be equal to the two Sides F B, B C, each
to each; but the Baſe D F has been proved equal to
the Baſe F C. Therefore the Angle F A D is * equal * 8. 1.
to the Angle F B C: Moreover, A G has been proved
equal to B H; but F B alſo, is equal to A F, there-
fore the two Sides F A, A G, are equal to the two
Sides F B, B H; and the Angle F A G is equal to
the Angle F B H, as has been demonſtrated; where-
fore the Baſe G F † is equal to the Baſe F H. Again,
becauſe G E has been proved equal to E H, and E F
is common, the two Sides G E, E F, are equal to the
two Sides H E, E F; but the Baſe H F is equal to the
Baſe F G; therefore the Angle G E F is * equal to
the Angle H E F; and ſo both the Angles G E F,
H E F, are Right Angles: Therefore F E makes
Right Angles with G H, which is any how drawn
thro' E. After the ſame manner we demonſtrate, that
E F is at Right Angles to all Right Lines that are
drawn in the Plane to it; but a Right Line is * at * Def. 3. of
Right Angles to a Plane, when it is at Right Angles this.
to all Right Lines drawn to it in the Plane. There-
fore F E is at Right Angles to a Plane drawn thro' the

Right Line A B, C D. Wherefore, *if to two Right Lines, cutting one another, a third ſtands at Right Angles in the common Section, it ſhall be alſo at Right Angles to the Plane drawn thro' the ſaid Lines*; which was to be demonſtrated.

PROPOSITION V.

Theorem.

If to three Right Lines, touching one another, a fourth ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane.

LET the Right Line A B ſtand at Right Angles, in the Point of Contact B, to the three Right Lines B C, B D, B E. I ſay, B C, B D, B E, are in one and the ſame Plane.

For, if they are not, let B D, B E, be in one Plane, and B C above it; and let the Plane paſſing thro' A B, B C, be produced, and it will * make the common Section, with the other Plane, a ſtrait Line, which let be B F; then three Right Lines A B, B C, B F, are in one Plane drawn thro' A B, B C: And ſince A B ſtands at Right Angles to B D and B E, it ſhall be † at Right Angles to a Plane drawn thro' B E, D B; and ſo A B ſhall make ‡ Right Angles with all Right Lines touching it that are in the ſame Plane: But B F, being in the ſaid Plane, touches it; wherefore the Angle A B F is a Right Angle: But the Angle A B C (by the *Hyp.*) is alſo a Right Angle; therefore the Angle A B F is equal to the Angle A B C, and they are both in the ſame Plane, which cannot be; and ſo the Right Line B C, is not above the Plane paſſing thro' B E and B D. Wherefore the three Right Lines B C, B D, B E, are in one and the ſame Plane. Therefore, *if to three Right Lines, touching one another, a fourth ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane*; which was to be demonſtrated.

* 3 of this.

† 4 of this.

‡ Def. 3.

PROPOSITION VI.

Theorem.

If two Right Lines be perpendicular to one and the ſame Plane, thoſe Right Lines are parallel to one another.

LET two Right Lines A B, C D, be perpendicular to one and the ſame Plane. I ſay, A B is parallel to C D.

For, let them meet the Plane in the Points B, D; and join the Right Line B D, to which let D E be drawn in the ſame Plane, at Right Angles, make D E equal to A B; and join B E, A E, A D.

Then becauſe A B is at Right Angles to the afore-
ſaid Plane, it ſhall be * at Right Angles to all Right * Def. 3. of this.
Lines, touching it, drawn in the Plane; but A B
touches B D, B E, which are in the ſaid Plane; there-
fore each of the Angles A B D, A B E, is a Right
Angle. So, for the ſame Reaſon, likewiſe, is each of
the Angles C D B, C D E, a Right Angle. Then,
becauſe A B is equal to D E, and B D is common;
the two Sides A B, B D, ſhall be equal to the two
Sides E D, D B; but they contain Right Angles:
Therefore the Baſe A D is † equal to the Baſe B E. † 4. 1.
Again, becauſe A B is equal to D E, and A D to B E;
the two Sides A B, B E, are equal to the two Sides
E D, D A; but A E, their Baſe, is common; where-
fore the Angle A B E is ‡ equal to the Angle E D A. ‡ 8. 1.
But A B E is a Right Angle; therefore E D A is alſo
a Right Angle; and ſo E D is perpendicular to D A:
But it is alſo perpendicular to B D and D C; there-
fore E D is at Right Angles, in the Point of Contact,
to three Right Lines B D, D A, D C: Wherefore
theſe three laſt Right Lines are * in one Plane. But * 5 of this.
B D, D A, are in the ſame Plane as A B is; for every
Triangle is † in the ſame Plane; therefore it is neceſ- † 2 of this.
ſary, that A B, B D, D C, be in one Plane. But
both the Angles A B D, B D C, are Right Angles;
wherefore A B is ‡ parallel to C D. Therefore, *if* ‡ 28. 1.
*two Right Lines be perpendicular to one and the ſame
Plane, thoſe Right Lines are parallel to one another*;
which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If there be two parallel Lines, and any Points be taken in both of them, the Right Line joining thoſe Points ſhall be in the ſame Plane as the Parallels are.

LET AB, CD, be two parallel Right Lines, in which are taken any Points E, F. I ſay, a Right Line joining the Points E, F, is in the ſame Plane as the Parallels are.

For, if it be not, let it be elevated above the ſame, if poſſible, as EGF, thro' which let ſome Plane be drawn, whoſe Section, with the Plane in which the
* 3 *of this.* Parallels are, let * be the Right Line EF; then the two Right Lines EGF, EF, will include a Space,
† *Ax.* 10. 1. which is † abſurd: Therefore a Right Line, drawn from the Point E to the Point F, is not elevated above the Plane; and, conſequently, it muſt be in that paſſing thro' the Parallels AB, CD. Wherefore, *if there be two parallel Lines, and any Points be taken in both of them, the Right Line joining thoſe Points ſhall be in the ſame Plane as the Parallels are*; which was to be demonſtrated.

PROPOSITION VIII.

THEOREM.

If there be two parallel Right Lines, one of which is perpendicular to ſome Plane; then ſhall the other be perpendicular to the ſame Plane.

See the Fig. of Prop. VI. LET AB, CD, be two parallel Right Lines, one of which, as AB, is perpendicular to ſome Plane. I ſay, the other, CD, is alſo perpendicular to the ſame Plane.

For, let AB, CD, meet the Plane in the Points B, D; and let BD be joined; then AB, CD, BD, are
* 7 *of this.* * in one Plane. Let DE be drawn in the other Plane,

Plane, at Right Angles to B D, and make D E equal
to A B; and join B E, A E, A D: Then, ſince A B
is perpendicular to the Plane, it will * be perpendicular * *Def.* 3.
to all Right Lines touching it, that are drawn in
the ſame Plane; therefore each of the Angles A B D,
A B E, is a Right Angle. And ſince the Right Line
B D falls on the Right Lines A B, C D; the Angles
A B D, C D B, ſhall be † equal to two Right Angles: † 29. 1.
Therefore the Angle C D B is alſo a Right Angle;
and ſo C D is perpendicular to D B. And ſince A B
is equal to D E, and B D is common; the two Sides
A B, B D, are equal to the two Sides E D, D B. But
the Angle A B D is equal to the Angle E D B; for
each of them is a Right Angle; therefore the Baſe
A D is ‡ equal to the Baſe B E. Again, ſince A B is ‡ 4. 1.
equal to D E, and B E to A D; the two Sides A B,
B E, ſhall be equal to the two Sides E D, D A, each
to each: But the Baſe A E is common; wherefore
the Angle A B E is * equal to the Angle E D A: But * 8. 1.
the Angle A B E is a Right Angle; therefore E D A
is alſo a Right Angle, and E D is perpendicular to
D A: But it is likewiſe perpendicular to D B; therefore
E D ſhall alſo be † perpendicular to the Plane
paſſing thro' B D, D A, and, likewiſe, ſhall be ‡ at † 4 *of this.*
Right Angles to all Right Lines, drawn in the ſaid ‡ *Def.* 3.
Plane that touch it. But D C is in the Plane paſſing
thro' B D, D A, becauſe A B, B D, are * in that * 2 *of this.*
Plane; and D C is † in the ſame Plane that A B and † 7 *of this.*
B D are in; wherefore E D is at Right Angles to
D C, and ſo C D is at Right Angles to D E, as alſo
to D B. Therefore, C D *ſtands at Right Angles, in
the common Section* D, *to two Right Lines* D E, D B,
*mutually cutting one another; and, accordingly, is at
Right Angles to the Plane paſſing thro'* D E, D B;
which was to be demonſtrated.

PROPOSITION IX.

THEOREM.

Right Lines that are parallel to the ſame Right Line, not being in the ſame Plane with it, are alſo parallel to each other.

LET both the Right Lines A B, C D, be parallel to the Right Line E F, not being in the ſame Plane with it. I ſay, A B is parallel to C D.

For aſſume any Point G in E F, from which Point G let G H be drawn, at Right Angles to E F, in the Plane paſſing thro' E F, A B: Alſo let G K be drawn at Right Angles to E F in the Plane paſſing thro' E F, C D: Then, becauſe E F is perpendicular to G H
* 4 of this. and G K, the Line E F ſhall alſo be * at Right Angles to a Plane paſſing thro' both G H and G K: But
† 6 of this. E F is parallel to A B; therefore A B is † alſo at Right Angles to the Plane paſſing thro' H G K. For the ſame Reaſon, C D is alſo at Right Angles to the Plane paſſing thro' H G K; and therefore A B, and C D, will be both at Right Angles to the Plane paſſing thro' H G K. But if two Right Lines be at
* 6 of this. Right Angles to the ſame Plane, they ſhall be * parallel to each other; therefore A B is parallel to C D. And ſo, *Right Lines that are parallel to the ſame Right Line, not being in the ſame Plane with it, are alſo parallel to each other*; which was to be demonſtrated.

PROPOSITION X.

THEOREM.

If two Right Lines, touching one another, be parallel to two other Right Lines, touching one another, but not in the ſame Plane, thoſe Right Lines contain equal Angles.

LET two Right Lines A B, B C, touching one another, be parallel to two Right Lines D E, E F, touching one another, but not in the ſame Plane. I ſay, the Angle A B C is equal to the Angle D E F.

For, take B A, B C, E D, E F, equal one to another,
and join A D, C F, B E, A C, D F: Then,
becauſe B A is equal and parallel to E D, the Right
Line A D ſhall alſo be * equal and parallel to B E. * 33. 1.
For the ſame Reaſon, C F will be equal and parallel
to B E; therefore A D, C F, are both equal and parallel
to B E. But Right Lines that are parallel to
the ſame Right Line, not being in the ſame Plane with
it, will be † parallel to each other. Therefore A D † 9 of this.
is parallel and equal to C F; but A C, D F, join
them; wherefore A C is ‡ equal and parallel to D F. ‡ 33. 1.
And becauſe the two Right Lines A B, B C, are equal
to the two Right Lines D E, E F, and the Baſe A C
equal to the Baſe D F; therefore the Angle A B C
will be * equal to the Angle D E F. Whence, *if* * 8. 1.
*two Right Lines, touching one another, be parallel to two
other Right Lines touching one another, but not in the
ſame Plane, thoſe Right Lines contain equal Angles*;
which was to be demonſtrated.

PROPOSITION XI.

PROBLEM.

From a Point given above a Plane, to draw a Right Line perpendicular to that Plane.

LET A be the Point given, above the given Plane B H. It is required to draw a Right Line from the Point A, perpendicular to the Plane B H.

Let a Right Line B C be any how drawn in the Plane
B H; and let A D be drawn * from the Point A, * 12. 1.
perpendicular to B C; then if A D be perpendicular
to the Plane B H, the Thing required is already done;
but, if not, let D E be drawn in the Plane from the
Point D, at Right Angles to B C; and let A F be
drawn * from the Point A, perpendicular to D E:
Laſtly, thro' F draw G H, parallel to B C.

Then, becauſe B C is perpendicular to both D A
and D E, B C will alſo be † perpendicular to a Plane † 4 of this.
paſſing thro' E D, D A. But G H is parallel to B C;
and if there are two Right Lines parallel, one of
which is at Right Angles to ſome Plane, then ſhall
the other be ‡ at Right Angles to the ſame Plane: ‡ 8 of this.
Wherefore G H is at Right Angles to the Plane paſ-

ſing thro' E D, D A, and ſo is * perpendicular to all the Right Lines, in the ſame Plane that touch it. But A F, which is in the Plane paſſing thro' E D and D A, doth touch it. Therefore G H is perpendicular to A F; and ſo A F is perpendicular to G H; but A F, likewiſe, is perpendicular to D E; therefore A F is perpendicular to both H G, D E. But if a Right Line ſtands at Right Angles to two Right Lines, in their common Section, that Line will be † at Right Angles to the Plane paſſing thro' theſe Lines. Therefore A F is perpendicular to the Plane drawn thro' E D, G H; that is, to the given Plane B H. Therefore, A F *is drawn from the given Point* A, *above the given Point* B H, *perpendicular to the ſaid Plane*; which was to be done.

* *Def.* 3.

† 4 *of this.*

PROPOSITION XII.

PROBLEM.

To erect a Right Line perpendicular to a given Plane, from a Point given therein.

LET A be a given Point in a given Plane M N. It is required to draw a Right Line from the Point A, at Right Angles to the Plane M N.

Let ſome Point B be ſuppoſed above the given Plane, from which let B C be drawn * perpendicular to the ſaid Plane; and let A D be drawn † from A, parallel to B C.

* 11 *of this.*

† 31. 1.

Then, becauſe A D, C B, are two parallel Right Lines, one of which, *viz.* B C, is perpendicular to the Plane M N; the other, A D, ſhall be ‡ alſo perpendicular to the ſame Plane. Therefore, *a Right Line is erected perpendicular to a given Plane, from a Point given thereon*; which was to be done.

‡ 8 *of this.*

PROPOSITION XIII.

THEOREM.

Two Right Lines cannot be erected at Right Angles to a given Plane, from a Point therein given.

FOR, if it is poſſible, let two Right Lines A B, A C, be erected perpendicular to a given Plane on the ſame Side, at a given Point A, in the given Plane.

Then let a Plane be drawn thro' B A, A C, cutting the given Plane thro' A in the Right Line * D A E; * *3 of this.* but the Line D A E being in the given Plane, touches it; therefore the Right Lines A B, A C, D A E, are in one Plane: And becauſe C A is perpendicular to the given Plane, it ſhall alſo be † perpendicular to all † *Def. 3.* Right Lines drawn in that Plane, and touching it: Therefore the Angle C A E is a Right Angle. For the ſame Reaſon, B A E is alſo a Right Angle; wherefore the Angle C A E is equal to B A E, and they are both in one Plane; which is abſurd. Therefore, *two Right Lines cannot be erected at Right Angles, to a given Plane, from a Point therein given*; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

Thoſe Planes, to which the ſame Right Line is perpendicular, are parallel to each other.

LET the Right Line A B be perpendicular to each of the Planes D C, E F. I ſay, theſe Planes are parallel.

For, if they be not, let them be produced till they meet each other, and let the Right Line G H be the common Section, in which take any Point K, and join A K, B K. Then, becauſe A B is perpendicular to the Plane E F, it ſhall alſo be perpendicular to the Right Line B K, being in the Plane E F produced; wherefore the Angle A B K is a Right Angle. And, for the ſame Reaſon, B A K is alſo a Right Angle.

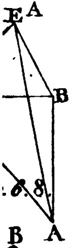
And ſo the two Angles A B K, B A K, of the Tri-
angle A B K, are equal to two Right Angles, which
* 17. 1. is * impoſſible: Therefore the Planes C D, E F, be-
ing produced, will not meet each other; and ſo are
neceſſarily parallel. Therefore, *thoſe Planes, to which
the ſame Right Line is perpendicular, are parallel to each
other*; which was to be demonſtrated.

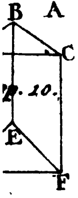
PROPOSITION XV.

THEOREM.

If two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the ſame Plane with them; the Planes drawn thro' thoſe Right Lines are parallel to each other.

LET two Right Lines A B, B C, touching one ano-
ther, be parallel to two Right Lines D E, E F,
touching one another, but not in the ſame Plane with
them. I ſay, the Planes paſſing thro' A B, B C, and
D E, E F, being produced, will not meet each other.
* 11 of this. For, let B G be drawn * from the Point B, perpen-
dicular to the Plane paſſing thro' D E, E F, meeting
the ſame in the Point G; and thro' G let G H be
‡ 31. 1. drawn ‡ parallel to E D, and G K parallel to E F;
then, becauſe B G is perpendicular to the Plane paſ-
* Def. 3. ſing thro' D E, E F, it ſhall alſo make * Right An-
gles with all Right Lines that touch it, and are in the
ſame Plane. But G H and G K, which are both in
the ſame Plane, touch it; therefore each of the An-
gles B G H, B G K, is a Right Angle. And ſince
B A is parallel to G H, the Angles G B A, B G H,
* 29. 1. are * equal to two Right Angles: But B G H is a
Right Angle; wherefore G B A ſhall alſo be a Right
Angle; and ſo B G is perpendicular to B A. For the
ſame Reaſon, G B is alſo perpendicular to B C;
therefore ſince a Right Line B G ſtands at Right An-
gles to two Right Lines B A, B C, mutually cutting
‡ 4 of this. each other; B G ſhall alſo be ‡ at Right Angles to
the Plane drawn thro' B A, B C. But it is perpendi-
cular to the Plane drawn thro' D E, E F; therefore


E
A
B
8.
A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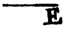

E

D
6.

BG is perpendicular to both the Planes drawn thro' AB, BC, and DE, EF. But thoſe Planes to which the ſame Right Line is perpendicular, are * parallel; therefore the Plane drawn thro' AB, BC, is parallel to the Plane drawn thro' DE, EF. Wherefore, *if two Right Lines, touching one another, be parallel to two Right Lines touching one another, and not being in the ſame Plane with them; the Planes drawn thro' thoſe Right Lines are parallel to each other*; which was to be demonſtrated.

* 14 of this.

PROPOSITION XVI.

THEOREM.

If two parallel Planes are cut by another Plane, their common Sections will be parallel.

LET two parallel Planes AB, CD, be cut by any Plane EFGH; and let their common Sections be EF, GH. I ſay, EF is parallel to GH.

For, if it is not parallel, EF, GH, being produced, will meet each other either on the Side FH, or EG. Firſt, let them be produced on the Side FH, and meet in K; then, becauſe EFK is in the Plane AB, all Points taken in EFK will be in the ſame Plane. But K is one of the Points that is in EFK; therefore K is in the ſame Plane AB. For the ſame Reaſon K is alſo in the Plane CD; wherefore the Planes AB, CD, will meet each other. But they do not meet, ſince they are ſuppoſed parallel; therefore the Right Lines EF, GH, will not meet on the Side FH. After the ſame manner it is proved, that they will not meet, if produced, on the Side EG. But Right Lines, that will neither Way meet each other, are parallel; therefore EF, is parallel to GH. *If*, therefore, *two parallel Planes are cut by any other Plane, their common Sections will be parallel*; which was to be demonſtrated.

PRO-

PROPOSITION XVII.

THEOREM.

If two Right Lines are cut by parallel Planes, they ſhall be cut in the ſame Proportion.

LET two Right Lines A B, C D, be cut by parallel Planes G H, K L, M N, in the Points A, E, B, C, F, D. I ſay, as the Right Line A E is to the Right Line E B, ſo is C F to F D.

For, let A C, B D, A D, be joined; let A D meet the Plane K L in the Point X; and join E X, X F. Then, becauſe two parallel Planes K L, M N, are cut by the Plane E B D X, their common Sections
* 16 of this. E X, B D, are * parallel. For the ſame Reaſon, becauſe two parallel Planes G H, K L, are cut by the Plane A X F C, their common Sections A C, F X, are parallel; and becauſe E X is drawn parallel to the Side B D of the Triangle A B D, it ſhall be, as
† 2. 6. A E is to E B, ſo is † A X to X D. Again, becauſe X F is drawn parallel to the Side A C of the Triangle A D C, it ſhall be †, as A X is to X D, ſo is C F to F D. But it has been proved, as A X is to X D, ſo
‡ 11. 5. is A E to E B. Therefore, as A E is to E B, ſo is ‡ C F to F D. Wherefore, *if two Right Lines are cut by parallel Planes, they ſhall be cut in the ſame Proportion*; which was to be demonſtrated.

PROPOSITION XVIII.

THEOREM.

If a Right Line be perpendicular to ſome Plane, then all Planes paſſing thro' that Line will be perpendicular to the ſame Plane.

LET the Right Line A B be perpendicular to the Plane C L. I ſay, all Planes that paſs thro' A B, are likewiſe perpendicular to the Plane C L.

For, let a Plane D E paſs thro' the Right Line A B, whoſe common Section, with the Plane C L, is the Right Line C E; and take ſome Point F in C E; from which let F G be drawn in the Plane D E, perpendicular to the Right Line C E: Then, becauſe A B is perpendicular to the Plane C L, it ſhall alſo be * perpendicular to all the Right Lines which touch it, and are in the ſame Plane: Wherefore it is perpendicular to C E; and, conſequently, the Angle A B F is a Right Angle: But G F B is likewiſe a Right Angle; therefore A B is parallel to F G. But A B is at Right Angles to the Plane C L; therefore F G will be † at Right Angles to that ſame Plane. But one Plane is perpendicular to another, when the Right Lines drawn in one of the Planes, perpendicular to the common Section of the Planes, are ‡ perpendicular to the other Plane. But F G is drawn in one Plane D E, perpendicular to the common Section C E of the Planes, and it has been proved to be perpendicular to the Plane C L: In like manner any other Line in the Plane D E, drawn perpendicular to C E, is proved to be perpendicular to the Plane C L. Therefore the Plane D E is at Right Angles to the Plane C L. After the ſame manner we demonſtrate, that all Planes paſſing thro' the Right Line A B, are perpendicular to the Plane C L. Therefore, *if a Right Line be perpendicular to ſome Plane, then all Planes, paſſing thro' that Line, will be perpendicular to the ſame Plane*; which was to be demonſtrated.

* *Def.* 3.

† 8 *of this.*

‡ *Def* 4. *of this.*

PROPOSITION XIX.

THEOREM.

If two Planes, cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plane.

LET two Planes A B, B C, cutting each other, be perpendicular to ſome third Plane, and let their common Section be B D. I ſay, B D is perpendicular to the ſaid third Plane, which let be A D C.

For, if poſſible, let B D not be perpendicular to the third Plane; and from the Point D let D E be drawn in the Plane A B, perpendicular to A D; and let D F be drawn, in the Plane B C, perpendicular to C D: Then, becauſe the Plane A B is perpendicular to the third Plane, and D E is drawn in the Plane A B, perpendicular to their common Section A D; D E ſhall

* Def. 4. ſhall be * perpendicular to the third Plane ADC. In like manner we prove, that DF alſo is perpendicular to the ſaid Plane; wherefore two Right Lines ſtand at Right Angles to this third Plane, on the ſame Side, at † 13 of this. the ſame Point D; which is † abſurd: Therefore, to this third Plane cannot be erected any Right Lines perpendicular at D, and on the ſame Side, except BD, the common Section of the Planes AB, BC: Whereſore DB is perpendicular to the third Plane. *If thereſore, two Planes, cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plane*; which was to be demonſtrated.

PROPOSITION XX.

Theorem.

If a ſolid Angle be contained under three plane Angles, any two of them, howſoever taken, are greater than the third.

LET the ſolid Angle A be contained under three plane Angles BAC, CAD, DAB. I ſay, any two of the Angles BAC, CAD, DAB, are greater than the third, howſoever taken.

For, if the Angles BAC, CAD, DAB, be equal, it is evident, that any two, howſoever taken, are greater than the third; but, if not, let BAC be the greater, * 23. 1. and make * the Angle BAE, at the Point A, with the Right Line AB, in a Plane paſſing thro' BA, AC, equal to the Angle DAB; make AE equal to AD; thro' E draw BEC, cutting the Right Lines AB, AC, in the Points B, C; and join DB, DC: Then, becauſe DA is equal to AE, and AB is common, the two Sides DA, AB, are equal to the two Sides AE, AB; but the Angle DAB, is equal to the † 4. 1. Angle BAE; therefore the Baſe DB is † equal to the Baſe BE: And ſince the two Sides DB, DC, are greater than BC, and DB has been proved equal to BE; therefore the remaining Side DC ſhall be greater than the remaining Side EC; and ſince DA is equal to AE, and AC is common, and the Baſe DC greater ‡ 25. 1. than the Baſe EC; the Angle DAC ſhall be ‡ greater than the Angle E A C. But, from the Conſtruction, the Angle D A B, is equal to the Angle B A E; wherefore the Angles D A B, D A C, are greater than the Angle B A C. After this manner we demonſtrate, if any two other Angles be taken, that they are greater than the third. Therefore, *if a ſolid Angle be contained under three plane Angles, any two of them, howſoever taken, are greater than the third*; which was to be demonſtrated.

PROPOSITION XXI.

Theorem.

Every ſolid Angle is contained under plane Angles, together, leſs than four Right ones.

LET A be a ſolid Angle, contained under plane Angles B A C, C A D, D A B. I ſay, the Angles B A C, C A D, D A B, are leſs than four Right Angles.

For, take any Points, B, C, D, in each of the Lines A B, A C, A D; and join B C, C D, D B: Then, becauſe the ſolid Angle at B is contained under three plane Angles C B A, A B D, C B D; any two of theſe are * greater than the third: Therefore the Angles * 20 *of this.* C B A, A B D, are greater than the Angle C B D. For the ſame Reaſon, the Angles B C A, A C D, are greater than the Angle B C D; and the Angles C D A, A D B, greater than the Angle C D B. Wherefore the ſix Angles C B A, A B D, B C A, A C D, C D A, A D B, are greater than the three Angles C B D, B C D, C D B. But the three Angles C B D, B C D, C D B, are † equal to two Right Angles; † 32. 1. wherefore the ſix Angles C B A, A B D, B C A, A D C, D C A, A D B, are greater than two Right Angles. And ſince the three Angles of each of the Triangles A B C, A C B, A D B, are equal to two Right Angles, the nine Angles of thoſe Triangles C B A, B C A, B A C, A C D, C A D, A D C, A D B, A B D, D A B, are equal to ſix Right Angles; ſix of which Angles C B A, B C A, A C D, A D C, A D B, A B D, are greater than two Right Angles. Therefore the three other Angles B A C, C A D, D A B, which contain the ſolid Angle, will be leſs than four Right Angles. Whereſore, *every ſolid Angle is contained under plane Angles, together, leſs than four Right ones*; which was to be demonſtrated.

PROPOSITION XXII.

THEOREM.

If there be three plane Angles, whereof two, any how taken, are greater than the third, and the Right Lines that contain them be equal; then it is poſſible to make a Triangle of the Right Lines joining the equal Right Lines which form the Angles.

LET A B C, D E F, G H K, be given plane Angles, any two whereof are greater than the third; and let the equal Right Lines A B, B C, D E, E F, G H, H K, contain them; and let A C, D F, G K, be joined. I ſay, it is poſſible to make a Triangle of A C, D F, G K; that is, any two of them, howſoever taken, are greater than the third.

For, if the Angles at B, E, H, are equal; then A C,
* 4. 1. D F, G K, will be * equal, and any two of them greater than the third; but, if not, let the Angles at B, E, H, be unequal; and let the Angle B be greater than either of the others at E, or H: Then the Right
† 24. 1. Line A C will be † greater than either D F, or G K; and it is manifeſt, that A C, together with either D F, or G K, is greater than the other. I ſay, likewiſe, that
‡ 23. 1. D F, G K, together, are greater than A C. For make ‡, at the Point B, with the Right Line A B, the Angle A B L equal to the Angle G H K; and make B L equal to either A B, B C, D E, E F, G H, H K; and join A L, C L. Then, becauſe the two Sides A B, B L, are equal to the two Sides G H, H K, each to each; and they contain equal Angles; the Baſe A L ſhall be equal to the Baſe G K. And ſince the Angle E and H are greater than the Angle A B C, the Angle G H K is equal to the Angle A B L, and therefore the other Angle at E ſhall be greater than the Angle L B C. And ſince the two Sides L B, B C, are equal to the two Sides D E, E F, each to each, and the Angle

D E F is greater than the Angle L B C, the Baſe D F ſhall be * greater than the Baſe L C. But G K has been proved equal to A L; therefore D F, G K, are greater than A L, L C: But A L, L C, are greater than A C; wherefore D F, G K, ſhall be much greater than A C. Therefore, any two of the Right Lines A C, D F, G K, howſoever taken, are greater than the third: And ſo, *a Triangle may be made of* A C, D F, G K; which was to be demonſtrated. * 24. 1.

PROPOSITION XXIII.

PROBLEM.

To make a ſolid Angle of three plane Angles, whereof any two, howſoever taken, are greater than the third; but theſe three Angles muſt be leſs than four Right Angles.

LET A B C, D E F, G H K, be three plane Angles given, whereof any two, howſoever taken, are greater than the third; and let the ſaid three Angles be leſs than four Right Angles; it is required to make a ſolid Angle of three plane Angles equal to A B C, D E F, G H K.

Let the Right Lines A B, B C, D E, E F, G H, H K, be cut off equal; and join A C, D F, G K; then it is poſſible to make * a Triangle of three Right Lines equal to A C, D F, G K: And ſo † let the Triangle L M N be made, ſo that A C be equal to L M, and D F to M N, and G K to L N; and let the Circle L M N be deſcribed ‡ about the Triangle, whoſe Centre let be X, which will be either within the Triangle L M N, or on one Side thereof, or without the ſame. * 22 of this. † 22. 1. ‡ 5. 4.

Firſt, let it be within, and join L X, M X, N X: I ſay, A B is greater than L X. For, if this be not ſo, A B ſhall be either equal to L X, or leſs. Firſt, let it be equal; then, becauſe A B is equal to L X, and alſo to B C, L X ſhall be equal to B C: But L X is equal to X M; therefore the two Sides A B, B C, are equal to the two Sides L X, X M, each to each; but the Baſe A C is put equal to the Baſe L M; wherefore the Angle A B C ſhall be * equal to the Angle L X M. * 8. 1.

For the ſame Reaſon, the Angle DEF is equal to the Angle MXN, and the Angle GHK to the Angle NXL; therefore the three Angles ABC, DEF, GHK, are equal to the three Angles LXM, MXN, NXL. But the three Angles LXM, MXN, NXL *Cor.* 15. 1. * are equal to four Right Angles; and ſo the three Angles ABC, DEF, GHK, ſhall be equal to four Right Angles: But they are put leſs than four Right Angles, which is abſurd; therefore AB is not equal to LX. I ſay alſo, it is neither leſs than LX; for, if this be poſſible, make XO equal to BA, and XP to BC, and join OP: Then, becauſe AB is equal to BC, XO ſhall be equal to XP; and the remaining Part OL, equal to the remaining Part PM; and † 2. 6. ſo LM is † parallel to OP, and the Triangle LMX is equiangular to the Triangle OPX: Wherefore ‡ 4. 6. XL is ‡ to LM, as XO is to OP; and (by Alternation) as XL is to XO, ſo is LM to OP. But LX is greater than XO; therefore LM ſhall alſo be greater than OP. But LM is put equal to AC; wherefore AC ſhall be greater than OP: And ſo, becauſe the two Right Lines AB, BC, are equal to the two Right Lines OX, XP, and the Baſe AC greater than * 25. 1. the Baſe OP; the Angle ABC will be * greater than the Angle OXP. In like manner we demonſtrate, that the Angle DEF is greater than the Angle MXN, and the Angle GHK than the Angle NXL; therefore the three Angles ABC, DEF, GHK, are greater than the three Angles LXM, MXN, NXL: But the Angles ABC, DEF, GHK, are put leſs than four Right Angles; therefore the Angles LXM, MXN, NXL, ſhall be leſs by much than four Right † *Cor.* 15. 1. Angles, and alſo equal † to four Right Angles; which is abſurd: Wherefore AB is not leſs than LX. It has alſo been proved not to be equal to it; therefore it muſt neceſſarily be greater. On the Point X raiſe ‡ 12 *of this.* ‡ XR, perpendicular to the Plane of the Circle LMN, whoſe Length let be ſuch, that the Square thereof be equal to the Exceſs by which the Square of AB exceeds the Square of LX; and let RL, RM, RN, be joined: Becauſe RX is perpendicular to the Plane * *Def.* 3. of the Circle LMN, it ſhall alſo be * perpendicular to LX, MX, NX: And becauſe LX is equal to

XM,

X M, and X R is common, and at Right Angles to them, the Baſe L R ſhall be * equal to the Baſe R M. * 4. 1. For the ſame Reaſon, R N is equal R L, or R M; therefore three Right Lines R L, R M, R N, are equal to each other. And becauſe the Square of X R is equal to the Exceſs by which the Square of A B exceeds the Square of L X, the Square of A B will be equal to the Squares of L X, X R, together: But the Square of R L is † equal to the Squares of L X, X R, for L X R is a † 47. 1. Right Angle; therefore the Square of A B will be equal to the Square of R L; and ſo A B is equal to R L. But B C, D E, E F, G H, H K, are every one of them equal to A B; and R N, or R M, equal to R L; wherefore A B, B C, D E, E F, G H, H K, are each equal to R L, R M, or R N: And ſince the two Sides R L, R M, are equal to the two Sides A B, B C; and the Baſe L M is put equal to the Baſe A C; the Angle L R M ſhall be ‡ equal to the Angle A B C. For ‡ 8. 1. the ſame Reaſon the Angle M R N is equal to the Angle D E F, and the Angle L R N equal to the Angle G H K: Therefore, *a ſolid Angle is made at* R *of three plane Angles* L R M, M R N, L R N, *equal to three plane Angles given* A B C, D E F, G H K.

Now, let the Centre of the Circle X be in one Side of the Triangle, *viz.* in the Side M N; and join X L: I ſay, again, that A B is greater than L X. For, if it be not ſo, A B will be either equal, or leſs than L X. Firſt, let it be equal; then the two Sides A B, B C, are equal to the two Sides M X, L X, that is, they are equal to M N: But M N is put equal to D F; therefore D E, E F, are equal to D F, which is * impoſſible; * 20. 1. therefore A B is not equal to L X. In like manner we prove, that it is neither leſſer; for the Abſurdity will much more evidently follow. Therefore A B is greater than L X. And if the Square of R X be made equal to the Exceſs by which the Square of A B exceeds the Square of L X, and R X be raiſed at Right Angles to the Plane of the Circle, *the Problem may be done* in like manner as before.

Laſtly, Let the Centre X of the Circle be without the Triangle L M N, and join L X, M X, N X: I ſay, A B is greater than L X. For, if it be not, it muſt either be equal, or leſs. Firſt, let it be equal; then

then the two Sides A B, B C, are equal to the two
Sides M X, X L, each to each; and the Baſe A C is
equal to the Baſe M L; therefore the Angle A B C is
† 8. 1. † equal to the Angle M X L. For the ſame Reaſon,
the Angle G H K is equal to the Angle L X N; and ſo
the whole Angle M X N is equal to the two Angles
A B C, G H K: But the Angles A B C, G H K, are
greater than the Angle D E F; therefore the Angle
M X N is greater than D E F: But becauſe the two
Sides D E, E F, are equal to the two Sides M X, X N,
and the Baſe D F is equal to the Baſe M N; the Angle
M N X ſhall be † equal to the Angle D E F: But it
has been proved greater, which is abſurd; therefore A B
is not equal to L X. Moreover, we will prove, that
it is not leſs; wherefore it ſhall be neceſſarily greater.
And if, again, X R be raiſed at Right Angles to the
Plane of the Circle, and made equal to the Side of
that Square by which the Square of A B exceeds the
Square of L X; the Problem will be determined.
Now, I ſay, A B is not leſs than L X: For, if it is
poſſible that it can be leſs, make X O equal to A B,
and X P equal to B C, and join O P; then, becauſe A B
is equal to B C, X O ſhall be equal to X P; and the
remaining Part O L equal to the remaining Part P M;
* 2. 6. therefore L M is * parallel to P O, and the Triangle
L M X equiangular to the Triangle P X O: Where-
† 4. 6. fore, as † X L is to L M, ſo is X O to O P; and (by
Alternation) as L X is to X O, ſo is L M to O P: But
L X is greater than X O; therefore L M is greater than
O P; but L M is equal to A C; wherefore A C ſhall be
greater than O P: And ſo, becauſe the two Sides A B,
B C, are equal to the two Sides O X, X P, each to
each; and the Baſe A C is greater than the Baſe O P;
‡ 25. 1. the Angle A B C ſhall be ‡ greater than the Angle
O X P. So, likewiſe, if X R be taken equal to X O,
or X P, and O R be joined, we prove, that the Angle
G H K is greater than the Angle O X R. At the Point
X, with the Right Line L X, make the Angle L X S
equal to the Angle A B C, and the Angle L X T equal
to the Angle G H K, and X S, X T, each equal to X O;
and join O S, O T, S T; then, becauſe the two Sides
A B, B C, are equal to the two Sides O X, X S, and
the Angle A B C is equal to the Angle O X S, the Baſe

A C, that is, L M, ſhall be equal to the Baſe O S. For the ſame Reaſon, L N is alſo equal to O T: And ſince the two Sides M L, L N, are equal to the two Sides O S, O T; and the Angle M L N, or P O R, is evidently greater than the Angle S O T; the Baſe M N ſhall be greater than the Baſe S T. But M N is equal to D F; therefore D F ſhall be greater than S T. Wherefore, becauſe the two Sides E F, D E, are equal to the two Sides S X, X T; and the Baſe D F is greater than the Baſe S T; the Angle D E F ſhall be greater than the Angle S X T. But the Angle S X T is equal to the Angles A B C, G H K; therefore the Angle D E F is greater than the Angles A B C, G H K: But it is alſo leſs (by *Hyp.*) which is abſurd; and conſequently, A B is not leſs than L X. And ſo, *a ſolid Angle may be made of three plane Angles that have the neceſſary Limitations*; which was to be done.

PROPOSITION XXIV.

THEOREM.

If a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof are equal Parallelograms.

LET the Solid C D G H be contained under parallel Planes A C, G F, B G, C E, F B, A E. I ſay, the oppoſite Planes thereof are equal Parallelograms.

For, becauſe the parallel Planes B G, C E, are cut by the Plane A C, their common Sections are * parallel; wherefore A B is parallel to C D. Again, becauſe the two parallel Planes B F, A E, are cut by the Plane A C, their common Sections are parallel; therefore A D is parallel to B C. But A B has been proved to be parallel to C D; wherefore A C ſhall be a Parallelogram. After the ſame manner we demonſtrate, that C E, F G, G B, B F, and A E, are Parallelograms.

Let A H, D F, be joined: Then, becauſe A B is parallel to D C, and B H to C F; the Lines A B, B H, touching each other, ſhall be parallel to the Lines D C, C F, touching each other, and not being in the ſame Plane; wherefore they ſhall † contain equal Angles: And ſo the Angle A B H is equal to the Angle D C F.

* 16 *of this.*

† 10 *of this.*

‡ 34. 1. D C F. And ſince the two Sides A B, B H, are ‡ equal
to the two Sides D C, C F, and the Angle A B H equal
* 4. 1. to the Angle D C F; the Baſe A H ſhall be * equal to
the Baſe D F, and the Triangle A B H equal to the
Triangle D F C: And ſince the Parallelogram B G is
‡ 41. 1. ‡ double to the Triangle A B H, and the Parallelo-
gram C E to the Triangle D C F; the Parallelogram
B G ſhall be equal to the Parallelogram C E. In like
manner we demonſtrate, that the Parallogram A C is
equal to the Parallelogram G F, and the Parallelogram
A E equal to the Parallelogram B F. *If, therefore, a
Solid be contained under ſix parallel Planes, the oppoſite
Planes thereof are Parallelograms*; which was to be
demonſtrated.

Coroll. It follows, from what has been now demonſtrated, that, if a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof are ſimilar and equal, becauſe each of the Angles are equal, and the Sides about the equal Angles are proportional.

PROPOSITION XXV.

THEOREM.

If a ſolid Parallelepipedon be cut by a Plane, parallel to oppoſite Planes, then, as Baſe is to Baſe, ſo is Solid to Solid.

LET the ſolid Parallelepipedon A B C D be cut by a Plane Y E, parallel to the oppoſite Plane R A, D H. I ſay, as the Baſe E F Φ A is to the Baſe E H C F, ſo is the Solid A B F Y to the Solid E G C D.

For, let A H be both Ways produced, and make
H M, M N, *&c.* equal to E H, and A K, K L, *&c.*
equal to A E; and let the Parallelograms L O, K Φ,
H X, M S, as likewiſe the Solids L P, K R, H Ω,
M T, be compleated: Then becauſe the Right Lines
L K, K A, A E, are equal; the Parallelograms L O,
* 36. 1. K Φ, A F, ſhall * alſo equal; as likewiſe the Pa-
† 14 *of this.* rallelograms K Ξ, K B, A G: And, moreover, † the
Parallelograms L Ψ, K P, A R, for they are oppoſite to
one another. For the ſame Reaſons, the Parallelograms
E C, H X, M S, alſo, are equal to each other; as alſo the

Parallelograms H G, H I, I N; and ſo are the Parallelograms D H, M Ω, N T: Therefore three Planes of the Solid L P are equal to three Planes of the Solid K R, or A Y, each to each; and the Planes oppoſite to theſe are equal to them: Therefore the three Solids L P, K R, R Y, will be equal * to each other. For the ſame Reaſon, the three Solids E D, H Ω, M T, are equal to each other; therefore the Baſe L F is of the Multiple of the Baſe A F, as the Solid L Y is of the Solid A Y. For the ſame Reaſon, the Baſe N F is the ſame Multiple of the Baſe H F, as the Solid N Y is of the Solid E D; and if the Baſe L F be equal to the Baſe N F, the Solid L Y ſhall be equal to the Solid N Y; and if the Baſe L F exceeds the Baſe N F, the Solid L Y ſhall exceed the Solid N Y; and, if it be leſs, leſs: Wherefore, becauſe there are four Magnitudes, *viz.* the two Baſes A F, F H, and the two Solids A Y, E D, whoſe Equimultiples are taken, to wit, the Baſe L F, and the Solid L Y; and the Baſe N F, and the Solid N Y: And ſince it is proved, if the Baſe L F exceeds the Baſe N F, then the Solid L Y will exceed the Solid N Y; if equal, equal; and if leſs, leſs: Therefore as the Baſe A F is to the Baſe F H, ſo is * the Solid A Y to the Solid E D. Wherefore, *if a ſolid Parallelepipedon be cut by a Plane, parallel to oppoſite Planes; then, as Baſe is to Baſe, ſo ſhall Solid be to Solid*; which was to be demonſtrated.

* Def. 10. of this.

* Def. 5. 5.

PROPOSITION XXVI.

PROBLEM.

At a Right Line given, and at a Point given in it, to make a ſolid Angle equal to a ſolid Angle given.

LET A B be a Right Line given, A a given Point in it, and D a given ſolid Angle contained under the plane Angles E D C, E D F, F D C; it is required to make a ſolid Angle at the given Point A, in the given Right Line A B, equal to the given ſolid Angle D.

Aſſume any Point F in the Right Line D F, from which let F G be drawn * perpendicular to the Plane

* 11 of this.

paſſing thro' E D, D C, meeting the ſaid Plane in the
† 23. 1. Point G, and join D G; make † the Angles B A L, B A K, at the given Point A, with the Right Line A B, equal to the Angles E D C, E D G.

Laſtly, make A K equal to D G, and at the Point
‡ 12 of this. K erect ‡ H K, at Right Angles to the Plane paſſing thro' B A L; and make K H equal to G F; and join H A. I ſay, the ſolid Angle at A, which is contained under the three plane Angles B A L, B A H, H A L, is equal to the ſolid Angle at D, which is contained under the plane Angles E D C, E D F, F D C: For let the equal Right Lines A B, D E, be taken; and join H B, K B, F E, G E: Then, becauſe F G is perpendicular
* Def. 3. of this. to the Plane paſſing thro' E D, D C, it ſhall be * perpendicular to all the Right Lines touching it, that are in the ſaid Plane: Wherefore both the Angles F G D, F G E, are Right Angles. For the ſame Reaſon, both the Angles H K A, H K B, are Right Angles; and becauſe the two Sides K A, A B, are equal to the two Sides G D, D E, each to each, and contain equal Angles,
† 4. 1. the Baſe B K ſhall be † equal to the Baſe E G: But K H is alſo equal to G F; and they contain Right Angles; therefore, H B ſhall be † equal to F E. Again, becauſe the two Sides A K, K H, are equal to the two Sides D G, G F, and they contain Right Angles; the Baſe A H ſhall be equal to the Baſe D F: But A B is equal to D E; therefore the two Sides H A, A B, are equal to the two Sides F D, D E. But the Baſe H B is equal to the Baſe F E; and ſo the Angle B A H will be
‡ 8. 1. ‡ equal to the Angle E D F: For the ſame Reaſon, the Angle H A L is equal to the Angle F D C: For ſince, if A L be taken equal to D C; and K L, H L, G C, F C, be joined; the whole Angle B A L is equal to the whole Angle E D C; and the Angle B A K, a Part of the one, is put equal to the Angle E D G, a Part of the other; the Angle K A L, remaining, will be equal to the Angle G D C remaining. And becauſe the two Sides K A, A L, are equal to the two Sides G D, D C, and they contain equal Angles; the Baſe K L will be equal to the Baſe G C: But K H is equal to G F; wherefore the two Sides L K, K H, are equal to the two Sides C G, G F: But they contain Right Angles; therefore the Baſe H L will be equal to the

Baſe F C. Again, becauſe the two Sides H A, A L, are equal to the two Sides F D, D C; and the Baſe H L is equal to the Baſe F C; the Angle H A L will be equal to the Angle F D C: But the Angle B A L was made equal to the Angle E D C: Therefore a *ſolid Angle is made equal to a ſolid Angle given*; which was to be done.

PROPOSITION XXVII.

THEOREM.

Upon a Right Line given, to deſcribe a Parallelepipedon, ſimilar, and in like manner ſituate, to a ſolid Parallepipedon.

LET A B be a Right Line, and C D a given ſolid Parallelepipedon; it is required to deſcribe a ſolid Parallepipedon upon the given Right Line A B, ſimilar, and alike ſituate, to the given ſolid Parallelepipedon C D.

Make a ſolid Angle at the given Point A, in the Right Line A B, * contained under the Angles B A H, * 26 *of this.* H A K, K A B; ſo that the Angle B A H may be equal to the Angle E C F, the Angle B A K to the Angle E C G, and the Angle H A K to the Angle G C F; and make, as E C is to C G, ſo B A † to A K; and † 12. 6. as G C to C F, ſo K A to A H: Then (by Equality of Proportion) as E C is to C F, ſo ſhall B A be to A H: Compleat the Parallelogram B H, and the Solid A L; then, becauſe it is, as E C is to G C, ſo is A B to A K; *viz.* the Sides about the equal Angles E C G, B A K, proportional; the Parallelogram K B ſhall be ſimilar to the Parallelogram G E. Alſo, for the ſame Reaſon, the Parallelogram K H ſhall be ſimilar to the Parallelogram G F, and the Parallelogram H B to the Parallelogram F E: Therefore three Parallelograms of the Solid A L, are ſimilar to three Parallelograms of the Solid C D. But theſe three Parallelograms are ‡ equal and ſimilar to their three oppoſite ones; there- ‡ *Cor.* 20. *of this.* fore the whole Solid A L will be ſimilar to the whole Solid C D; and ſo, *a ſolid Parallelepipedon* A L *is deſcribed upon the given Right Line* A B, *ſimilar and alike ſituate, to the given ſolid Parallelepipedon* C D; which was to be done.

PRO-

PROPOSITION XXVIII.

Theorem.

If a ſolid Parallelepipedon be cut by a Plane paſſing thro' the Diagonals of two oppoſite Planes, that Solid will be biſected by the Plane.

LET the ſolid Parallelepipedon A B be cut by the Plane C D E F, paſſing thro' the Diagonals C F, D E, of two oppoſite Planes. I ſay, the Solid A B is biſected by the Plane C D E F.

* 34. 1. For, becauſe the Triangle C G F is * equal to the Triangle CBF, and the Triangle ADE to the Triangle † 24 of this. D E H, and the Parallelogram C A to † the Parallelogram B E, for it is oppoſite to it; and the Parallelogram G E to the Parallelogram C H; the Priſm contained by the two Triangles C G F, A D E, and the three Parallelograms G E, A C, C E, is equal to the Priſm contained under the two Triangles C F B, DEH, ‡ Def. 10. of this. and the three Parallelograms CH, BE, CE; for ‡ they are contained under Planes equal in Number and Magnitude. Therefore, *the whole Solid* A B *is biſected by the Plane* C D E F; which was to be demonſtrated.

PROPOSITION XXIX.

Theorem.

Solid Parallelepipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, and whoſe inſiſtent Lines are in the ſame Right Lines, are equal to one another.

LET the ſolid Parallelepipedons C M, B F, be conſtituted upon the ſame Baſe A B, with the ſame Altitude, whoſe inſiſtent Lines A F, A G, L M, L N, C D, C E, B H, B K, are in the ſame Right Lines F N, D K. I ſay, the Solid C M is equal to the Solid B F.

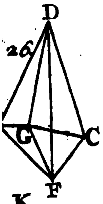
* 34. 1. For, becauſe C H, C K, are both Parallelograms, C B ſhall be * equal to D H, or E K; wherefore D H is

C

K

N

N
R
T
13.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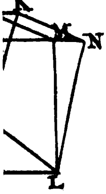
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is equal to E K. Let E H, which is common, be taken away, then the Remainder D E will be equal to the Remainder H K, and ſo the Triangle D E C is † equal to the Triangle H K B, and the Parallelogram D G equal to the Parallelogram H N; for the ſame Reaſon the Triangle A F G is equal to the Triangle L M N. Now the Parallelogram C F ‡ is equal to the Parallelogram B M, and the Parallelogram C G to the Parallelogram B N, for they are oppoſite. Therefore the Priſm contained under the two Triangles A F G, D E C, and the three Parallelograms C F, D G, C G, is * equal to the Priſm contained under the two Triangles L M N, H B K, and the three Parallelograms B M, H N, B N. Let the common Solid, whoſe Baſe is the Parallelogram A B, oppoſite to the Parallelogram G E H M, be added, then the whole ſolid Parallelepipedon C M is equal to the whole ſolid Parallelepipedon B F. Therefore, *ſolid Parallelepipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, and whoſe inſiſtent Lines are in the ſame Right Lines, are equal to one another*; which was to be demonſtrated.

† 8. 1.
‡ 24 of this.
* Def. 10. of this.

PROPOSITION XXX.

THEOREM.

Solid Parallelepipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, whoſe inſiſtent Lines are not placed in the ſame Right Lines, are equal to one another.

LET there be ſolid Parallelepipedons C M, C N, having equal Altitudes, and ſtanding on the ſame Baſe A B, and whoſe inſiſtent Lines A F, A G, L M, L N, C D, C E, B H, B K, are not in the ſame Right Lines. I ſay, the Solid C M is equal to the Solid C N.

For let N K, D H, be produced, and G E, F M, be drawn, meeting each other in the Points R, X: Let alſo F M, G E, be produced to the Points O, P; and join A X, L O, C P, B R. The Solid C M, whoſe Baſe is the Parallelogram A C B L, being oppoſite to the Parallelogram F D H M, is * equal to the Solid C O, whoſe

* 29 of this.

whoſe Baſe is the Parallelogram A C B L, being oppoſite to X P R O, for they ſtand upon the ſame Baſe A C B L; and the inſiſtent Lines A F, A X, L M, L O, C D, C P, B H, B R, are in the ſame right Lines F O, D R: But the Solid C O, whoſe Baſe is the Parallelogram A C B L, being oppoſite to X P R O, is * equal to the Solid C N, whoſe Baſe is the Parallelogram A C B L, being oppoſite to G E K N; for they ſtand upon the ſame Baſe A C B L, and their inſiſtent Lines A G, A X, C E, C P, L N, L O, B K, B R, are in the ſame Right Lines G P, N R: Wherefore the Solid C M ſhall be equal to the Solid C N. Therefore, *ſolid Parallelepipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, whoſe inſiſtent Lines are not placed in the ſame Right Lines, are equal to one another*; which was to be demonſtrated. * 29 of this.

PROPOSITION XXXI.

Theorem.

Solid Parallelepipedons, being conſtituted upon equal Baſes, and having the ſame Altitude, are equal to one another.

LET A E, C F, be ſolid Parallelepipedons, conſtituted upon the equal Baſes A B, C D, and having the ſame Altitude. I ſay, the Solid A E is equal to the Solid C F.

First, Let H K, B E, A G, L M, O P, D F, C z, R S, be at Right Angles to the Baſes A B, C D; let the Angle A L B not be equal to the Angle C R D, and produce C R to T, ſo that R T be equal to A L; then make the Angle T R Y, at the Point R, in the Right Line R T, equal * to the Angle A L B; make R Y equal to L B; draw X Y, thro' the Point Y, * parallel to R T ‡, and compleat the Parallelogram R X, and the Solid ψ Y. Therefore, becauſe the two Sides T R, R Y, are equal to the two Sides A L, L B, and they contain equal Angles; the Parallelogram R X ſhall be equal and ſimilar to the Parallelogram H L. And again, becauſe A L is equal to R T, and L M to R S, and they contain equal Angles, the Parallelogram R ψ ſhall

* 23. 1.

‡ 31. 1.

ſhall be equal and ſimilar to the Parallelogram A M. For the ſame Reaſon the Parallelogram L E is equal and ſimilar to the Parallelogram S Y; therefore three Parallelograms of the Solid A E are equal and ſimilar to three Parallelograms of the Solid ϒ Y; and ſo the three oppoſite ones of one Solid are † alſo equal and † *24 of this.* ſimilar to the three oppoſite ones of the other: Therefore the whole ſolid Parallelepipedon A E is equal to the whole ſolid Parallelepipedon ϒ Y. Let D R, X Y, be produced, and meet each other in the Point Ω, and let T Q be drawn thro' T * parallel to D Ω, and pro- * *31. 1.* duce T Q, O D, till they meet in V, and compleat the Solids Ωϒ, R I: Then the Solid ϒΩ, whoſe Baſe is the Parallelogram R ϒ, and ΩΓ is that oppoſite to it, is ‡ ‡ *29 of this.* equal to the Solid ϒ Y, whoſe Baſe is the Parallelogram R ϒ, and Y Φ is that oppoſite to it, for they ſtand upon the ſame Baſe R ϒ, have the ſame Altitude, and their inſiſtent Lines R Ω, R Y, T Q, T X, S Z, S N, ϒΓ, ϒΦ, are in the ſame Right Lines Ω X, Z Φ; but the Solid ϒ Y is equal to the Solid A E; and ſo A E is equal to the Solid ϒΩ. Again, becauſe the Parallelogram R Y X T is equal to the Parallelogram Ω T, for it ſtands on the ſame Baſe R T, and between the ſame Parallels R T, Ω X; and the Parallelogram R Y X T is equal to the Parallelogram C D, becauſe it is alſo equal to A B; the Parallelogram Ω T is equal to the Parallelogram C D, and D T is ſome other Parallelogram: Therefore, as the Baſe C D is to the Baſe D T, ſo is Ω T to T D. And becauſe the ſolid Parallelepipedon C I is cut by the Plane R F, being parallel to two oppoſite Planes; it ſhall be *, as the Baſe C D is to the Baſe D T, ſo is * *25 of this.* the Solid C F to the Solid R I. For the ſame Reaſon, becauſe the ſolid Parallelepipedon Ω I is cut by the Plane R ϒ, parallel to two oppoſite Planes; as the Baſe Ω T is to the Baſe D T, ſo ſhall * the Solid Ωϒ be to the Solid R I. But as the Baſe C D is to the Baſe D T, ſo is the Baſe Ω T to T D: Therefore, as the Solid C F is to the Solid R I, ſo is the ſolid Ωϒ to the Solid R I. And ſince each of the Solids C F, Ωϒ, has the ſame Proportion to the Solid R I, the Solid C F is equal to the Solid Ωϒ: But the Solid Ωϒ has been proved equal to the Solid A E; therefore

† 9. 5. ſore the Solid A E ſhall be † equal to the Solid C F. But, now let the inſiſtent Lines A G, H K, B E, L M, C N, O P, D F, R S, not be at Right Angles to the Baſes A B, C D. I ſay, again, that the Solid A E is equal to the Solid C F. Let there be drawn from the Points K, E, G, M, P, F, N, S, to the Plane wherein are the Baſes A B, C D, the Perpendiculars K Ξ, E T, G Y, M Φ, S I, F Ψ, N Ω, P X, meeting the Plane in the Points Ξ, T, Y, Φ, I, Ψ, Ω, X; and join Ξ T, Y Φ, Ξ Y, T Φ, X Ψ, X Ω, Ω I, Ψ I; then the Solid K Φ is equal to the Solid P I, for they ſtand on equal Baſes K M, P S, have the ſame Altitude, and the inſiſtent Lines are at Right Angles to the Baſes. But the Solid K Φ, is equal to the Solid A E, ‡ 29 *of this.* and the Solid P I to ‡ the Solid C F, ſince they ſtand upon the ſame Baſe, have the ſame Altitude, and their inſiſtent Lines are in the ſame Right Line: Therefore the Solid A E ſhall be equal to the Solid C F. Wherefore, *ſolid Parallepipedons, being conſtituted upon equal Baſes, and having the ſame Altitude, are equal to one another*; which was to be demonſtrated.

PROPOSITION XXXII.

Problem.

Solid Parallelepipedons, that have the ſame Altitude, are to each other as their Baſes.

LET A B, C D, be ſolid Parallelepipedons, that have the ſame Altitude. I ſay, they are to one another as their Baſes; that is, as the Baſe A E is to the Baſe C F, ſo is the Solid A B to the Solid C D.

For, apply a Parallelogram F H to the Right Line F G, equal to the Parallelogram A E; and compleat the ſolid Parallelepipedon G K upon the Baſe F H, having the ſame Altitude as C D has: Then the * 31 *of this.* Solid A B is * equal to the Solid G K, for they ſtand upon equal Baſes A E, F H, and have the ſame Altitude; and ſo, becauſe the ſolid Parallelepipedon C K is cut by the Plane D G, parallel to two oppoſite † 25 *of this.* Planes, it ſhall be †, as the Baſe H F is to the Baſe F C, ſo is the Solid H D to the Solid D C: But the

Baſe FH is equal to the Baſe AE, and the Solid AB to the Solid GK. Therefore, as the Baſe AE is to the Baſe CF, ſo is the Solid AB to the Solid CD. Wherefore, *ſolid Parallelepipedons, that have the ſame Altitude, are to each other as their Baſes*; which was to be demonſtrated.

PROPOSITION XXXIII.

Theorem.

Similar ſolid Parallelepipedons are to one another in the triplicate Proportion of their homologous Sides.

LET AB, CD, be ſimilar ſolid Parallelepipedons, and let the Side AE be homologous to the Side CF. I ſay, the Solid AB, to the Solid CD, hath a Proportion, triplicate of that, which the Side AE has to the Side CF.

For, produce AE, GE, HE, to EK, EL, EM; and make EK equal to CF, and EL to FN, and EM to FR; and let the Parallelogram KL, and likewiſe the Solid KO, be compleated: Then, becauſe the two Sides KE, EL, are equal to the two Sides CF, FN; and the Angle KEL equal to the Angle CFN (ſince the Angle AEG is equal to the Angle CFN, becauſe of the Similarity of the Solids AB, CD) the Parallelogram KL ſhall be ſimilar and equal to the Parallelogram CN. For the ſame Reaſon, the Parallelogram KM is equal and ſimilar to the Parallelogram CR, and the Parallelogram OE to DF; therefore three Parallelograms of the Solid KO are equal and ſimilar to three Parallelograms of the Solid CD: But thoſe three Parallelograms are * equal and ſimilar to the three oppoſite Parallelograms; therefore the whole Solid KO is equal and ſimilar to the whole Solid CD. Let the Parallelogram GK be compleated, as alſo the Solids EX, LP, upon the Baſes GK, KL, having the ſame Altitude as AB: And ſince, becauſe of the Similarity of the Solids AB and CD, it is, as AE is to CF, ſo is EG to FN; and ſo EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; it ſhall be, as

* *24 of this.*

† 1. 6. A E is to E K, ſo is † the Parallelogram A G to the
Parallelogram G K; but as G E is to E L, ſo is G K
to K L; and as H E is † to to E M, ſo is P E to K M:
Therefore, as the Parallelogram A G is to the Parallelogram
G K; ſo is G K to K L, and P E to K M.
‡ 32 *of this.* But as A G is to G K, ſo is ‡ the Solid A B to the Solid
E X; and as G K is to K L, ſo is the Solid E X to
the Solid P L; and as P E is to K M, ſo is the Solid
P L to the Solid K O: Therefore, as the Solid A B
* 11. 5. is to the Solid E X; ſo is * E X to P L, and P L to
K O: But if four Magnitudes be continually proportional,
† *Def.* 11. 5. the firſt to the fourth hath † a triplicate Proportion
of that which it has to the ſecond. Therefore,
alſo, the Solid A B to the Solid K O, hath a
triplicate Proportion of that which A B has to E X:
But as A B is to E X, ſo is the Parallelogram A G to
the Parallelogram G K; and ſo is the Right Line
A E to the Right Line E K: Wherefore the Solid
A B to the Solid K O, hath a Proportion triplicate of
that which A E has to E K. But the Solid K O is
equal to the Solid C D, and the Right Line E K
equal to the Right Line C F: Therefore, *the Solid*
A B, *to the Solid* C D, *has a Proportion triplicate of that
which the homologous Side* A E *has to the homologous Side*
C F; which was to be demonſtrated.

Coroll. From hence it is manifeſt, if four Right Lines be continually proportional, as the firſt is to the fourth, ſo is a ſolid Parallelepipedon deſcribed upon the firſt, to a ſimilar ſolid Parallelepipedon, alike ſituate, deſcribed upon the ſecond; becauſe the firſt to the fourth, has a Proportion triplicate of that which it has to the ſecond.

PROPOSITION XXXIV.

Theorem.

The Baſes and Altitudes of equal ſolid Parallelepipedons are reciprocally proportional; and thoſe ſolid Parallelepidedons, whoſe Baſes and Altitudes are reciprocally proportional, are equal.

LET A B, C D, be equal ſolid Parallelepipedons, I ſay, their Baſes and Altitudes are reciprocally proportional; that is, as the Baſe E H is to the Baſe N P, ſo is the Altitude of the Solid C D to the Altitude of the Solid A B.

Firſt, let the inſiſtent Lines A G, E F, L B, H K, C M, N X, O D, P R, be at Right Angles to their Baſes: I ſay, as the Baſe E H is to the Baſe N P, ſo is C M to A G. For, if the Baſe E H be equal to the Baſe N P, and the Solid A B is equal to the Solid C D; the Altitude C M ſhall alſo be equal to the Altitude A G: For if, when the Baſes E H, N P, are equal, the Altitudes A G, C M, are not ſo; then the Solid A B will not be equal to the ſolid C D, but it is put equal to it: Therefore the Altitude C M is not unequal to the Altitude A G, and ſo they are neceſſarily equal to one another; and, conſequently, as the Baſe E H is to the Baſe N P, ſo ſhall C M be to A G. But now let the Baſe E H be unequal to the Baſe N P, and let E H be the greater; then, ſince the Solid A B is equal to the Solid C D, C M is greater than A G; for, otherwiſe, it would follow, that the Solids A B, C D, are not equal, which are put ſuch: Therefore, make C T equal to A G, and compleat the ſolid Parallelepipedon V C upon the Baſe N P, having the Altitude C T: Then, becauſe the Solid A B is equal to the Solid C D, and V C is ſome other Solid; and ſince equal Magnitudes have * the ſame Proportion to the ſame Magni- * 7. 5.
tudes; it ſhall be, as the Solid A B is to the Solid C V, ſo is the Solid C D to the Solid C V: But as the Solid A B is to the Solid C V, ſo is † the Baſe E H to the † 32 *of this.*
Baſe N P; for A B, C V, are Solids having equal Altitudes: And as the Solid C D is to the Solid C V, ſo

‡ 25 *of this.* is ‡ the Baſe M P to the Baſe PT, and ſo is M C to C T: Therefore, as the Baſe E H is to the Baſe NP, ſo is MC to C T. But C T is equal to A G; wherefore, as the Baſe E H is to the Baſe N P, ſo is M C to A G: Therefore, *the Baſes and Altitudes of the equal ſolid Parallelepipedons* A B, C D, *are reciprocally proportional.*

Now, let the Baſes and Altitudes of the ſolid Parallelepipedons A B, C D, be reciprocally proportional; that is, let the Baſe E H be to the Baſe N P, as the Altitude of the Solid C D is to the Altitude of the Solid A B. I ſay, the Solid A B is equal to the Solid C D.

For, let again the inſiſtent Lines be at Right Angles to the Baſes; then, if the Baſe E H be equal to the Baſe N P, and E H is to N P as the Altitude of the Solid C D is to the Altitude of the Solid A B; the Altitude of the Solid C D ſhall be equal to the Altitude of the Solid A B. But ſolid Parallelepipedons, that ſtand upon equal Baſes, and have the ſame Altitude, are * equal to each other; therefore the Solid A B is equal to the Solid C D.

* 23 *of this.*

But now let the Baſe E H not be equal to the Baſe N P, and let E H be the greater; then the Altitude of the Solid C D is greater than the Altitude of the Solid A B; that is, C M is greater than A G: Again, put C T equal to A G, and compleat the Solid C V, as before; and then, becauſe the Baſe E H is to the Baſe N P, as M C is to A G, and A G is equal to C T; it ſhall be, as the Baſe E H is to the Baſe N P, ſo is M C to C T: But as the Baſe E H is to the Baſe N P, ſo is the Solid A B to the Solid C V; for the Solids ‡ A B, C V, have equal Altitudes; and as M C is to C T, ſo is the Baſe M P to the Baſe P T, and ſo the Solid C D to the Solid C V: Therefore as the Solid A B is to the Solid C V, ſo is the Solid C D to the Solid C V: But ſince each of the Solids A B, C D, has the ſame Proportion to C V; the Solid A B ſhall be equal to the Solid C D; whence, *the two ſolid Parallelepipedons* A B, C D, *whoſe Baſes and Altitudes are reciprocally proportional, are equal*; which was to be demonſtrated.

Now, let the inſiſtent Lines F E, B L, G A, K H, X N, D O, M C, R P, not be at Right Angles to the Baſes; and from the Points F, G, B, K, X, M, D, R, let there be drawn Perpendiculars to the Planes of the Baſes E H, N P, meeting the ſame in the Points S, T, Y, V, Q, Z, Ω, Φ, and compleat the Solids F V, X Ω. Then, I ſay, if the Solids A B, C D, be equal, their Baſes and Altitudes are reciprocally proportional; *viz.* as the Baſe E H is to the Baſe N P, ſo is the Altitude of the Solid C D to the Altitude of the Solid A B.

For, becauſe the Solid A B is equal to the Solid C D, and the Solid A B is * equal to the Solid B T; for they ſtand upon the ſame Baſe F K, and have the ſame Altitude; and the Solid D C is * equal to the Solid D Z, ſince they ſtand upon the ſame Baſe X R, and have the ſame Altitude; therefore the Solid B T ſhall be equal to the Solid D Z. But the Baſes and Altitudes of thoſe equal Solids, whoſe Altitudes are at Right Angles to their Baſes, are † reciprocally proportional; therefore as the Baſe F K is to the Baſe X R, ſo is the Altitude of the Solid D Z to the Altitude of the Solid B T. But the Baſe F K is equal to the Baſe E H, and the Baſe X R to the Baſe N P; wherefore, as the Baſe E H is to the Baſe N P, ſo is the Altitude of the Solid D Z to the Altitude of the Solid B T. But the Solids D Z, D C, have the ſame Altitude, and ſo have the Solids B T, B A; therefore the Baſe E H is to the Baſe N P, as the Altitude of the Solid D C is to the Altitude of the Solid A B; and ſo, *the Baſes and Altitudes of equal ſolid Parallelepipedons are reciprocally proportional.*

* 30 *of this.*

† *From what has been before proved.*

Again, let the Baſes and Altitudes of the ſolid Parallelepipedons A B, C D, be reciprocally proportional; *viz.* as the Baſe E H is to the Baſe N P, ſo let the Altitude of the Solid C D be to the Altitude of the Solid A B: I ſay, the Solid A B is equal to the Solid C D.

For, the ſame Conſtruction remaining, becauſe the Baſe E H is to the Baſe N P, as the Altitude of the Solid C D is to the Altitude of the Solid A B; and ſince the Baſe E H is equal to the Baſe F K, and N P to X R; it ſhall be, as the Baſe F K is to the Baſe X R, ſo is the Altitude of the Solid C D to the Altitude of the Solid A B. But the Altitudes of the Solids A B, B T, are the ſame; as alſo of the Solids C D, D Z; therefore the Baſe F K is to the Baſe X R, as the Altitude of the Solid D Z is to the Altitude of the Solid B T; wherefore the Baſes and Altitudes of the ſolid Parallelepipedons B T, D Z, are reciprocally proportional; but thoſe ſolid Parallelepipedons, whoſe Altitudes are at Right Angles to their Baſes, and the Baſes and Altitudes are reciprocally proportional, are equal to † each other. But the Solid B T is equal to the Solid B A, for they ſtand upon the ſame Baſe F K, and have the ſame Altitude; and the Solid D Z is alſo equal to the Solid D C, ſince they ſtand upon the ſame Baſe X R, and have the ſame Altitude: Therefore the Solid A B is equal to the Solid C D; whence *ſolid Parallelepipedons, whoſe Baſes and Altitudes are reciprocally proportional, are equal*; which was to be demonſtrated.

† *From what has been before proved.*

PROPOSITION XXXV.

THEOREM.

If there be two plane Angles equal, and from the Vertices of thoſe Angles two Right Lines be elevated above the Planes, in which the Angles are, containing equal Angles with the Lines firſt given, each to its correſpondent one; and if in thoſe elevated Lines any Points be taken, from which Lines be drawn perpendicular to the Planes in which the Angles firſt given are, and Right Lines be drawn to the Angles firſt given from the Points made by the Perpendiculars in the Planes; thoſe Right Lines will contain equal Angles with the elevated Lines.

LET B A C, E D F, be two equal Right-lined plane Angles, and from A and D, the Vertices of thoſe Angles, let two Right Lines, A G and D M, be elevated above the Planes of the ſaid Angles, making equal Angles with the Lines firſt given, each to its correſpondent one; *viz.* the Angle M D E equal to the Angle G A B, and the Angle M D F to the Angle G A C; and take any Points G and M in the Right Lines A G, D M; from which let G L and M N be drawn perpendicular to the Planes paſſing thro' B A C, E D F, meeting the ſame in the Points L and N; and join L A and N D. I ſay, the Angle G A L is equal to the Angle M D N.

Make A H equal to D M, and thro' H let H K be
drawn parallel to G L; but G L is perpendicalar to
the Plane paſſing thro' B A C; therefore H K ſhall be † † 8 *of this.*
alſo perpendicular to the Plane paſſing thro' B A C:
Draw from the Points K and N, to the Right Lines
A B, A C, D E, and D F, the Perpendiculars K B, K C,
N E, N F; and join H C, C B, M F, F E: Then, becauſe
the Square of H A is ‡ equal to the Squares of ‡ 47. 1.
H K, K A; and the Squares of K C and C A are †
equal to the Square of K A; the Square of H A ſhall
be equal to the Squares of H K, K C, and C A: But
the Square of H C is equal to the Squares of H K and
K C; therefore the Square of H A will be equal to the
Squares of H C and C A; and ſo the Angle H C A is ‡ ‡ 48. 1.
a Right Angle. For the ſame Reaſon, the Angle D F M
is alſo a Right Angle; therefore the Angle A C H is
equal to D F M: But the Angle H A C is alſo equal to
the Angle M D F; therefore the two Triangles
M D F, H A C, have two Angles of the one equal to
two Angles of the other, each to each, and one Side of
the one equal to one Side of the other; *viz.* that which
is ſubtended by one of the equal Angles; that is, the
Side H A equal to D M; and ſo the other Sides of the
one ſhall be * equal to the other Sides of the other, * 26. 1.
each to each: Wherefore A C is equal to D F. In
like manner we demonſtrate, that A B is equal to
D E: For, let H B, M E, be joined; then, becauſe
the Square of A H is equal to the Squares of A K
and K H; and the Squares of A B and B K are
equal to the Square of A K; the Squares of A B,
B K, and K H, will be equal to the Square of A H.
But the Square of B H is equal to the Squares of B K,
K H; for the Angle H K B is a Right Angle, becauſe
H K is perpendicular to the Plane paſſing thro' B A C;
therefore the Square of A H is equal to the Squares of
A B and B H: Wherefore the Angle A B H is † a † 48. 1.
Right Angle. For the ſame Reaſon the Angle D E M
is alſo a Right Angle; and the Angle B A H is equal
to the Angle E D M, for ſo it is put; and A H is
equal to D M; therefore A B is * alſo equal to D E: * 4. 1.
And ſo, ſince A C is equal to D F, and A B to D E;
the two Sides C A, A B, ſhall be equal to the two
Sides F D, D E: But the Angle B A C is equal to the
Angle F D E; therefore the Baſe B C is * equal to
the

the Baſe E F, the Triangle to the Triangle, and the other Angles to the other Angles: Wherefore the Angle A C B is equal to the Angle D F E. But the Right Angle A C K is equal to the Right Angle D F N; and therefore the remaining Angle B C K is equal to the remaining Angle E F N. For the ſame Reaſon, the Angle C B K is equal to the Angle F E N; and ſo, becauſe B C K and E F N are two Triangles, having two Angles equal to two Angles, each to each, and one Side equal to one Side, which is at the equal
* 26. 1. Angles; *viz.* B C equal to E F; therefore * they ſhall have the other Sides equal to the other Sides: Therefore C K is equal to F N. But A C is equal to D F; therefore the two Sides A C, C K, are equal to the two Sides D F, F N; and they contain Right Angles; conſequently, * the Baſe A K is equal to the Baſe D N. And ſince A H is equal to D M, the Square of A H ſhall be equal to the Square of D M: But the Squares of A K and K H are equal to the Square of A H; for the Angle A K H is a Right Angle; and the Squares of D N and N M are equal to the Square of D M, ſince the Angle D N M is a Right Angle; therefore the Squares of A K and K H are equal to the Squares of D N and N M; of which the Square of A K is equal to the Square of D N: Wherefore the Square of K H remaining is equal to the remaining Square of N M; and ſo the Right Line H K is equal to M N. And ſince the two Sides H A, A K, are equal to the two Sides M D, D N, each to each, and the Baſe K H has been proved equal to the Baſe
† 8. 1. N M, the Angle H A K, or G A L, ſhall be † equal to the Angle M D N; *which was to be demonſtrated.*

Coroll. From hence it is manifeſt, that if there be two Right-lined plane Angles equal, from whoſe Points equal Right Lines be elevated on the Planes of the Angles, containing equal Angles with the Lines firſt given, each to each; Perpendiculars drawn from the extreme Points of thoſe elevated Lines to the Planes of the Angles firſt given, are equal to one another.

PRO-

PROPOSITION XXXVI.

Theorem.

If three Right Lines be proportional, the ſolid Parallelepipedon made of them is equal to the ſolid Parallelepipedon made of the middle Line, if it be an equilateral one, and equiangular to the aforeſaid Parallelepipedon.

LET three Right Lines A, B, C, be proportional; *viz.* let A be to B, as B is to C. I ſay, the Solid made of A, B, C, is equal to the equilateral Solid made of B, equiangular to that made on A, B, C.

Let E be a ſolid Angle contained under the three plane Angles D E G, G E F, F E D; and make D E, G E, E F, each equal to B, and compleat the ſolid Parallelepipedon E K: Again, put L M equal to A, and at the Point L, at the Right Line L M, make * a ſolid Angle contained under the plane Angles N L X, X L M, M L N, equal to the ſolid Angle E; and make L N equal to B, and L X equal to C: Then, becauſe A is to B, as B is to C; and A is equal to L M; and B to L N, E F, E G, or E D; and C to L X; it ſhall be, as L M is to E F, ſo is G E to L X: And ſo the Sides about the equal Angles M L X, G E F, are reciprocally proportional. Wherefore the Parallelogram M X † is equal to the Parallelogram G F. And ſince the two plane Angles G E F, X L M, are equal, and the Right Lines L N, E D, being equal, are erected at the angular Points containing equal Angles with the Lines firſt given, each to each; the Perpendiculars drawn ‡ from the Points N and D, to the Planes drawn thro X L M, G E F, are equal one to another: Therefore the Solids L H, E K, have the ſame Altitude. But ſolid Parallelepipedons that have equal Baſes, and the ſame Altitude, are * equal to each other; therefore the Solid H L is equal to the Solid E K. But the Solid H L is that made of the three Right Lines A, B, C; and the Solid E K, that made of the Right Line B: Therefore, *if three Right Lines be proportional, the ſolid Parallelepipedon made of them is equal to the ſolid Parallelepipedon made of the middle Line, if it be an equilateral one,*

* 26 of this.

† 14. 6.

‡ Cor. 35. of this.

* 31 of this.

and equiangular to the aforeſaid Parallelepipedon; which was to be demonſtrated.

PROPOSITION XXXVII.

Theorem.

If four Right Lines be proportional, the ſolid Parallelepipedons ſimilar, and in like manner deſcribed from them, ſhall be proportional. And if the ſolid Parallelepipedons, being ſimilar, and alike deſcribed, be proportional, then the Right Lines they are deſcribed from, ſhall be proportional.

LET the four Right Lines A B, C D, E F, G H, be proportional; *viz.* let A B be to C D, as E F is to G H; and let the ſimilar and alike ſituate Parallelepipedons K A, L C, M E, N G, be deſcribed from them. I ſay, K A is to L C, as M E is to N G.

For, becauſe the ſolid Parallelepipedon K A is ſimilar to L C, therefore K A to L C ſhall be * a Proportion triplicate of that which A B has to C D. For the ſame Reaſon, the Solid M E to N G will have a triplicate Proportion of that which E F has to G H. But A B is to C D, as E F is to G H; therefore A K is to L C, as M E is to N G. And if the Solid A K be to the Solid L C, as the Solid M E is to the Solid N G; I ſay, as the Right Line A B is to the Right Line C D, ſo is the Right Line E F to the Right Line G H: For, becauſe A K to L C has † a Proportion triplicate of that which A B has to C D; and M E to N G has a Proportion triplicate of that which E F has to G H; and ſince A K is to L C, as M E is to N G; it ſhall be, as A B is to C D, ſo is E F to G H. Therefore, *if four Right Lines be proportional, the ſolid Parallelepipedons ſimilar, and in like manner deſcribed from them, ſhall be proportional. And if the ſolid Parallelepipedons, being ſimilar and alike deſcribed, be proportional, then the Right Lines they are deſcribed from, ſhall be proportional*; which was to be demonſtrated.

* 33 *of this.*

† 33 *of this.*

PRO-

PROPOSITION XXXVIII.

THEOREM.

If a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular ſhall fall in the common Section of the Planes.

LET the Plane C D be perpendicular to the Plane A B, let their common Section be A D, and let ſome Point E be taken in the Plane C D. I ſay, a Perpendicular, drawn from the Point E to the Plane A B, falls on A D.

For, if it does not, let it fall without the ſame, as E F, meeting the Plane A B in the Point F; and from the Point F let F G be drawn in the Plane A B, perpendicular to A D; this ſhall be * perpendicular to the Plane C D; and join E G: Then, becauſe F G is perpendicular to the Plane C D, and the Right Line E G, in the Plane C D, touches it; the Angle F G E ſhall be † a Right Angle. But E F is alſo at Right Angles to the Plane A B; therefore the Angle E F G is a Right Angle: And ſo, two Angles of the Triangle E F G are equal to two Right Angles; which is ‡ abſurd. Wherefore, a Right Line drawn from the Point E perpendicular to the Plane A B, does not fall without the Right Line A D; and ſo it muſt neceſſarily fall on it. Therefore, *if a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular ſhall fall in the common Section of the Planes*; which was to be demonſtrated.

* *Def.* 4. of this.
† *Def.* 3. of this.
‡ 17. 1.

PRO-

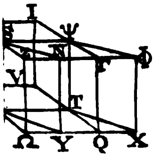
PROPOSITION XXXIX.

THEOREM.

If the Sides of the oppoſite Planes of a ſolid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of thoſe Planes, and the Diameter of the ſolid Parallelepipedon, ſhall divide each other into two equal Parts.

LET the Sides of CF, AH, the oppoſite Planes of the ſolid Parallelepipedon AF, be cut in Half in the Points K, L, M, N, X, O, P, R; and let the Planes KN, XR, be drawn thro' the Sections: Alſo, let YS be the common Section of the Planes, and DG the Diameter of the ſolid Parallelepipedon. I ſay, YS, DG, biſect each other; that is, YT is equal to TS, and DT to TG.

For, join DY, YE, BS, SG. Then, becauſe DX
is parallel to OE, the alternate Angles DXY, YOE,
* 29. 1. are * equal to one another. And becauſe DX is equal
to OE, and YX to YO, and they contain equal Angles,
† 4. 1. the Baſe DY ſhall be † equal to the Baſe YE, and
the Triangle DXY to the Triangle YOE, and the
other Angles equal to the other Angles: Therefore
the Angle XYD is equal to the Angle OYE; and ſo
‡ 14. 1. DYE is ‡ a Right Line. For the ſame Reaſon, BSG
is alſo a Right Line, and BS is equal to SG; then, becauſe
CA is equal and parallel to DB, as alſo to EG,
DB ſhall be equal and parallel to EG; and the Right
* 33. 1. Lines DE, GB, join them: Therefore DE is * parallel
to BG, and D, Y, G, S, are Points taken in each
of them; and DG, YS, are joined: Therefore DG,
† 7 *of this.* YS, are † in one Plane. And ſince DE is parallel to
* 29. 1. BG, the Angle EDT ſhall be * equal to the Angle
BGT, for they are alternate: But the Angle DTY is
‡ 15. 1. ‡ equal to the Angle GTS; therefore DTY, GTS,
are two Triangles, having two Angles of the one equal
to two Angles of the other, as likewiſe one Side of the
one equal to one Side of the other; *viz.* the Side DY
equal to the Side GS; for they are Halves of DE, BG;
* 26. 1. therefore* they ſhall have the other Sides of the one equal



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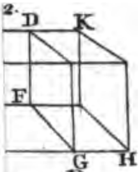
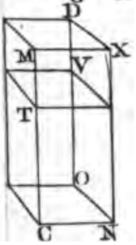
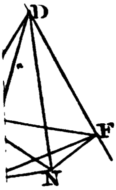
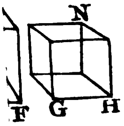
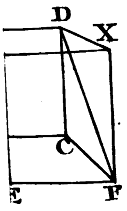










to the other Sides of the other; and ſo D T is equal to T G, and Y T to T S. Wherefore, *if the Sides of the oppoſite Planes of a ſolid Parallelepipedon, be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of theſe Planes, and the Diameter of the ſolid Parallelepipedon, ſhall divide each other into two equal Parts;* which was to be demonſtrated.

PROPOSITION XL.

THEOREM.

Of two triangular Priſms, one ſtanding on a Baſe which is a Parallelogram, and the other on a Triangle, if their Altitudes from theſe Baſes are equal, and the Parallelogram double to the Triangle, then thoſe Priſms are equal to each other.

LET ABCDEF, GHKLMN, be two Priſms of equal Altitude, the Baſe of one of which is the Parallelogram A F, and that of the other the Triangle G H K; and let the Parallelogram A F be double to the Triangle GHK. I ſay, the Priſm A B C D E F is equal to the Priſm G H K L M N.

For, compleat the Solids A X, G O. Then, becauſe the Parallelogram A F is double to the Triangle G H K; and ſince the Parallelogram H K is * double * 41. 1.
to the Triangle G H K; the Parallelogram A F ſhall be equal to the Parallelogram H K. But ſolid Parallelepipedons, that ſtand upon equal Baſes, and have the ſame Altitude, are † equal to one another; there- † 31 *of this.*
fore the Solid A X is equal to the Solid G O. But the Priſm A B C D E F is ‡ half the Solid A X; and the ‡ 28 *of this.*
Priſm G H K L M N is half the Solid G O; therefore the Priſm A B C D E F is equal to the Priſm G H K L M N. Wherefore, *if there be two triangular Priſms having equal Altitudes, the Baſe of one of which is a Parallelogram, and that of the other a Triangle; and if the Parallelogram be double to the Triangle, the ſaid Priſms ſhall be equal to each other;* which was to be demonſtrated.

The End *of the* Eleventh Book.

EUCLID's

ELEMENTS.

BOOK XII.

PROPOSITION I.

THEOREM.

Similar Polygons, inſcribed in Circles, are to one another as the Squares of the Diameters of the Circles.

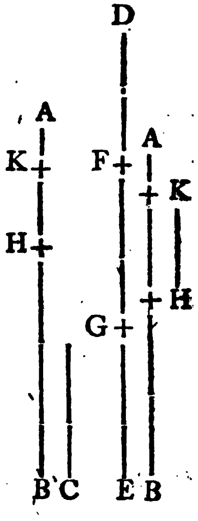
LET ABCDE, FGHKL, be Circles, wherein are inſcribed the ſimilar Polygons ABCDE, FGHKL; and let BM, GN, be Diameters of the Circles. I ſay, as the Square of BM is to the Square of GN, ſo is the Polygon ABCDE to the Polygon FGHKL.

For, join BE, AM, GL, FN. Then, becauſe the Polygon ABCDE is ſimilar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL: Therefore the two Triangles BAE, GFL, have one Angle of the one equal to one Angle of the other; *viz.* the Angle BAE equal to the Angle GFL, and the Sides about the equal Angles pro-
* 6. 6. portional. Wherefore the Triangle ABE is * equi-
angular to the Triangle FGL; and ſo the Angle AEB
is equal to the Angle FLG: But the Angle AEB is
† 21. 3. † equal to the Angle AMB, for they ſtand on the

ſame Circumference; and the Angle F L G is † equal † 21. 3. to the Angle F N G: Therefore the Angle A M B is equal to the Angle F N G. But the Right Angle B A M is ‡ equal to the Right Angle G F N; where- ‡ 31. 3. fore the other Angle ſhall be equal to the other Angle: And ſo the Triangle A M B is equiangular to the Triangle F G N; and, conſequently, as * B M is to * 4. 6. G N, ſo is B A to G F. But the Proportion of the Square of B M to the Square of G N, is duplicate of the Proportion of B M to G N; and the Proportion of the Polygon A B C D E to the Polygon F G H K L, is † duplicate of the Proportion of B A to G F: Where- † 20. 6. fore, as the Square of B M is to the Square of G N, ſo is the Polygon A B C D E to the Polygon F G H K L. Therefore, *ſimilar Polygons, inſcribed in Circles, are to one another as the Squares of the Diameters of the Circles*; which was to be demonſtrated.

LEMMA.

If there be two unequal Magnitudes propoſed, and from the greater be taken a Part greater than its Half; and if from what remains there be again taken a Part greater than half this Remainder; and again from this laſt Remainder a Part greater than its Half; and if this be done continually; there will remain at laſt a Magnitude that ſhall be leſs than the leſſer of the propoſed Magnitudes.



LET A B and C be two unequal Magnitudes, whereof A B is the greater: I ſay, if from A B be taken a greater Part than Half, and from the Part remaining there be again taken a Part greater than its

Half, and this be done continually, there will remain a Magnitude, at laſt, that ſhall be leſs than the Magnitude C.

For C being ſome Number of Times multiplied, will become greater than the Magnitude A B. Let it be multiplied, and let D E be a Multiple of C greater than A B; divide D E into Parts D F, F G, G E, each equal to C, and take B H, a Part greater than half of A B, from A B, and again from A H, the Part H K greater than half A H, and from A K a Part greater than half A K, and ſo on, until the Diviſions that are in A B, are equal in Number to the Diviſions in D E: Therefore let the Diviſions A K, K H, H B, be equal in Number to the Diviſions D F, F G, G E. Then, becauſe D E is greater than A B, and the Part E G taken from E D is leſs than half thereof, and the Part B H, greater than half of A B, is taken from it; the Part remaining, D G, ſhall be greater than the Part remaining, H A. Again, becauſe G D is greater than H A; and G F, being half of G D, is taken from the ſame; and H K, being greater than half H A, is taken from this likewiſe; the Part remaining, F D, ſhall be greater than the Part remaining, A K, But F D is equal to C; therefore C is greater than A K; and ſo the Magnitude A K is leſſer than C: Therefore, *the Magnitude* A K, *being the Part remaining of the Magnitude* A B, *is leſs than the leſſer propoſed Magnitude* C; which was to be demonſtrated. If the Halves of the Magnitudes ſhould have been taken, we demonſtrate this after the ſame manner. *This is the firſt Propoſition of the tenth Book.*

PROPOSITION II.

Theorem.

Circles are to each other as the Squares of their Diameters.

LET A B C D, E F G H, be Circles, whoſe Diameters are B D, F H. I ſay, as the Square of B D is to the Square of F H, ſo the Circle A B C D is to the Circle E F G H.

For, if it be not ſo, the Square of B D ſhall be to the Square of F H, as the Circle A B C D is to ſome Space either leſs or greater than the Circle E F G H. Firſt, let it be to a Space S, leſs than the Circle E F G H; and let the Square E F G H be deſcribed in the Circle.

This Square E F G H will be greater than half the Circle E F G H; becauſe, if we draw Tangents to the Circle thro' the Points E, F, G, H, the Square E F G H will be half that deſcribed about the Circle: But the Circle is leſs than the Square deſcribed about it; therefore the Square E F G H is greater than half the Circle E F G H. Let the Circumferences E F, F G, G H, H E, be biſected in the Points, K, L, M, N; and join E K, K F, F L, L G, G M, M H, H N, N E: Then each of the Triangles E K F, F L G, G M H, H N E, will be * greater than one half of the Segment of the * 41. 1. Circle it ſtands in; becauſe, if Tangents at the Circle be drawn thro' the Points K, L, M, N, and the Parallelograms that are on the Right Lines E F, F G, G H, H E, be compleated, each of the Triangles E K F, F L G, G M H, H N E, is half of each of the correſponding Parallelograms: But the Segment is leſs than the Parallelogram; wherefore each of the Triangles E K F, F L G, G M H, H N E, is greater than one half of the Segment of the Circle in which it ſtands: Therefore, if theſe Circumferences be again biſected, and Right Lines be drawn joining the Points of Biſection, and you do thus continually, there will at laſt remain Segments of the Circle, that ſhall be leſs than the Exceſs, by which the Circle E F G H exceeds the Space S. For it is demonſtrated, in the foregoing *Lemma*, that, if there be two unequal Magnitudes propoſed, and if from the greater a Part greater than half be taken, and again from the Part remaining a Part greater than half be taken, and you do thus continually, there will at laſt remain a Magnitude that will be leſs than the leſſer propoſed Magnitude. Let the Segments of the Circle E F G H, on the Right Lines E K, K F, F L, L G, G M, M H, H N, N E, be thoſe which are leſs than the Exceſs, whereby the Circle E F G H exceeds the Space S; and then the remaining Polygon E K F L G M H N ſhall be greater than the Space S. Alſo, deſcribe the Polygon

A X B O C P D R in the Circle A B C D, ſimilar to the Polygon E K F L G M H N. Wherefore, as the
* 1 *of this.* Square of B D is to the Square of F H, ſo is the Polygon A X B O C P D R to the * Polygon E K F L G M H N. But as the Square of B D is to the Square of F H, ſo is the Circle A B C D to the
‡ *From Hyp.* Space S ‡: Wherefore, as the Circle A B C D is to
† 11. 5. Space S, ſo is † the Polygon A X B O C P D R to the Polygon E K F L G M H N. But the Circle A B C D is greater than the Polygon in it; wherefore the
* 14. 5. Space S ſhall be * alſo greater than the Polygon
‡ *From Hyp.* E K F L G M H N: But it is leſs ‡ likewiſe; which is abſurd; therefore the Square of B D to the Square of F H, is not as the Circle A B C D to ſome Space leſs than the Circle E F G H. After the ſame manner we likewiſe demonſtrate, that the Square of F H to the Square of B D, is not as the Circle E F G H to ſome Space T leſs than the Circle A B C D. Laſtly, I ſay, the Square of B D to the Square of F G, is not as the Circle A B C D to ſome Space greater than the Circle E F G H: For, if it be poſſible, let it be ſo, and let the Space S be greater than the Circle E F G H: Then it ſhall be (by Inverſion) as the Square of F H is to the Square of B D, ſo is the Space S to the Circle A B C D. But, becauſe S is greater than the Circle E F G H, the Space S ſhall be to the Circle A B C D, as the Circle E F G H is to ſome Space T leſs than the Circle A B C D: Therefore, as the Square of F H is to the Square of B D,
* 11. 5. ſo is * the Circle E F G H to ſome Space T leſs than the Circle A B C D, which has been demonſtrated to be impoſſible; wherefore the Square of B D to the Square of F H, is not as the Circle A B C D to ſome Space greater than the Circle E F G H: But it alſo has been proved, that the Square of B D to the Square of F H, is not as the Circle A B C D to ſome Space leſs than the Circle E F G H: Wherefore, as the Square of B D is to the Square of F H, ſo ſhall the Circle A B C D be to the Circle E F G H. Wherefore, *Circles are to each other as the Squares of their Diameters*; which was to be demonſtrated.

PROPOSITION III.

Theorem.

Every Pyramid, having a triangular Baſe, may be divided into two Pyramids, equal and ſimilar to one another, having triangular Baſes, and ſimilar to the whole Pyramid; and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid.

LET there be a Pyramid, whoſe Baſe is the Triangle A B C, and Vertex the Point D. I ſay, the Pyramid A B C D may be divided into two Pyramids equal and ſimilar to one another, having triangular Baſes, and ſimilar to the Whole; and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid.

For, biſect A B, B C, C A, A D, D B, D C, in the Points E, F, G, H, K, L; and join E H, E G, G H, H K, K L, L H, E K, K F, F G: Then, becauſe A E is equal to E B, and A H to H D; E H ſhall be * parallel to D B; for the ſame Reaſon, H K alſo is parallel to A B; therefore H E B K is a Parallelogram; and ſo H K is † equal to E B: But E B is equal to A E; therefore A E ſhall be alſo equal to H K; but A H is equal to H D; wherefore the two Sides A E, A H, are equal to the two Sides K H, H D, each to each, and the Angle E A H is ‡ equal to the Angle K H D; wherefore the Baſe E H is * equal to the Baſe K D; and ſo the Triangle A E H is equal and ſimilar to the Triangle H K D. For the ſame Reaſon, the Triangle A H G ſhall alſo be equal and ſimilar to the Triangle H D L; and becauſe the two Right Lines E H, H G, touching each other, are parallel to the two Right Lines K D, D L, touching each other, and not in the ſame Plane with them, they ſhall contain † equal Angles. Therefore the Angle E H G is equal to the Angle K D L. Again, becauſe the two Sides E H, H G, are equal to the two Sides K D, D L, each to each; and the Angle E H G is equal to the Angle K D L; the Baſe E G ſhall be * equal to the Baſe K L; and therefore the Triangle E H G is equal and ſimilar to the Triangle K D L. For the ſame Reaſon, the Triangle

* 2. 6.
† 34. 1.
‡ 29. 1.
* 4. 1.
† 10. 11.
* 4. 1.

angle A E G is alſo equal and ſimilar to the Triangle
H K L; wherefore the Pyramid whoſe Baſe is the Triangle
‡ *Def.* 10. 11. A E G, and Vertex the Point H, ‡ is equal and
ſimilar to the Pyramid whoſe Baſe is the Triangle
H K L, and Vertex the Point D. And becauſe H K is
drawn parallel to the Side A B of the Triangle A D B,
* 2. 6. the Triangle A D B * ſhall be equiangular to the Triangle
D K H, and they have their Sides proportional;
therefore the Triangle A D B is ſimilar to the Triangle
D H K. And, for the ſame Reaſon, the Triangle D B C
is ſimilar to the Triangle D K L; and the Triangle
A D C to the Triangle D H L. And ſince the two
Right Lines B A, A C, touching each other, are parallel
to the two Lines K H, H L, touching each
other, not being in the ſame Plane with them, theſe
† 10. 11. ſhall contain equal Angles †; therefore the Angle
B A C is equal to the Angle K H L: And B A is to
A C, as K H is to H L; wherefore the Triangle
A B C is ſimilar to the Triangle H K L; and ſo the
Pyramid, whoſe Baſe is the Triangle A B C, and Vertex
the Point D, is ſimilar to the Pyramid, whoſe Baſe
is the Triangle H K L, and Vertex the Point D. But
the Pyramid, whoſe Baſe is the Triangle H K L, and
Vertex the Point D, has been proved ſimilar to the Pyramid
whoſe Baſe is the Triangle A E G, and Vertex
the Point H; therefore the Pyramid whoſe Baſe is the
Triangle A B C, and Vertex the Point D, is ſimilar
to the Pyramid whoſe Baſe is the Triangle A E G, and
Vertex the Point H: Wherefore both the Pyramids
A E G H, H K L D, are ſimilar to the whole Pyramid
A B C D. And becauſe B F is equal to F C, the Parallelogram
E B F G will be double to the Triangle
G F C; and ſince there are two Priſms of equal Altitude,
one of which has that Parallelogram for a Baſe,
and the other the Triangle, and the Parallelogram is
* 40. 11. double to the Triangle; thoſe Priſms will be * equal
to one another: Therefore the Priſm contained under
the two Triangles B K F, E H G, and the three Parallelograms
E B F G, E B K H, K H G F, is equal to
the Priſms contained under the two Triangles G F C,
H K L, and the three Parallelograms K F C L,
L C G H, H K F G. And it is manifeſt, that each of
thoſe Priſms, the Baſe of one of which is the Parallelogram
E B G F, and the oppoſite Baſe to that the

Right Line K H, and the Baſe of the other, the Triangle G F C, and the oppoſite Baſe to this the Triangle K L H, are greater than either of the Pyramids, whoſe Baſes are the Triangles A E G, H K L, and Vertices the Points H and D. For ſince, if the Right Lines E F, E H, be joined, the Priſm, whoſe Baſe is the Parallelogram E B F G, and the oppoſite Baſe to that the Right Line K H, is greater than the Pyramid, whoſe Baſe is the Triangle E B F, and Vertex the Point K. But the Pyramid whoſe Baſe is the Triangle E B F, and Vertex the Point K, is equal to the Pyramid whoſe Baſe is the Triangle A E G, and Vertex the Point H; for they are contained under equal and ſimilar Planes. Wherefore the Priſm whoſe Baſe is the Parallelograms E B F G, and the oppoſite Baſe to it the Right Line H K, is greater than the Pyramid whoſe Baſe is the Triangle A E G, and Vertex the Point H. But the Priſm, whoſe Baſe is the Parallelogram E B F G, and the oppoſite Baſe to it the Right Line H K, is equal to the Priſm whoſe Baſe is the Triangle G F C, and the oppoſite Baſe to this the Triangle H K L; and the Pyramid, whoſe Baſe is the Triangle A E G, and Vertex the Point H, is equal to the Pyramid whoſe Baſe is the Triangle H K L, and Vertex the Point D: Therefore the two Priſms aforeſaid are greater than the ſaid two Pyramids, whoſe Baſes are the Triangles A E G, H K L, and Vertices the Points H, D: And ſo the whole Pyramid, whoſe Baſe is the Triangle A B C, and Vertex the Point D, is divided into two equal Pyramids, ſimilar to each other, and to the Whole, and into two equal Priſms, which two Priſms, together, are greater than half of the whole Pyramid. Therefore, *every Pyramid, having a triangular Baſe, may be divided into two Pyramids, equal and ſimilar to one another, having triangular Baſes, and ſimilar to the whole Pyramid; and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid*; which was to be demonſtrated.

PROPOSITION IV.

Theorem.

If there are two Pyramids of the ſame Altitude, having triangular Baſes, and each of them be divided into two Pyramids, equal to one another, and ſimilar to the Whole, as alſo into two equal Priſms; and if, in like manner, each of the two Pyramids, made by the former Diviſion, be divided, and this be done continually; then, as the Baſe of one Pyramid is to the Baſe of the other Pyramid, ſo are all the Priſms that are in one Pyramid, to all the Priſms that are in the other Pyramid, being equal in Multitude.

LET there be two Pyramids of the ſame Altitude, having the triangular Baſes A B C, D E F, whoſe Vertices are the Points G, H; and let each of them be divided into two Pyramids, equal to one another, and ſimilar to the Whole, and into two equal Priſms; and if, in like manner, each of the Pyramids, made by the former Diviſion, be conceived to be divided, and this be done continually; I ſay, as the Baſe A B C is to the Baſe D E F, ſo are all the Priſms that are in the Pyramid A B C G, to all the Priſms that are in the Pyramid D E F H, being equal in Multitude.

For, ſince B X is equal to X C, and A L to L C, X L
* 2. 6. ſhall be * parallel to A B, and the Triangle A B C ſimilar to the Triangle L X C. For the ſame Reaſon, the Triangle D E F ſhall be alſo ſimilar to the Triangle R Q F: And becauſe B C is double to C X, and E F to F Q; it ſhall be, as B C is to C X, ſo is E F to F Q: And ſince there are deſcribed upon B C, C X, Right-lined Figures A B C, L X C, ſimilar and alike ſituate; and upon E F, F Q, Right-lined Figures D E F, R Q F, ſimilar and alike ſituate; therefore, as the Triangle
† 22. 6. B A C is to the Triangle L X C, ſo is † the Triangle D E F to the Triangle R Q F; and (by Alternation) as the Triangle A B C is to the Triangle D E F, ſo is the Triangle L X C to the Triangle R Q F. But as the Tri-
‡ 28. and angle L X C is to the Triangle R Q F, ſo is ‡ the Priſm,
32. 11. whoſe Baſe is the Triangle L X C, and the oppoſite Baſe

to that the Triangle O M N, to the Priſm, whoſe Baſe is the Triangle R Q F, and the oppoſite Baſe to that the Triangle S T Y; therefore, as the Triangle A B C is to the Triangle D E F, ſo is * the Priſm whoſe Baſe is the Triangle L X C, and the oppoſite Baſe to that the Triangle O M N, to the Priſm whoſe Baſe is the Triangle R Q F, and the oppoſite Baſe to that the Triangle S T Y: And becauſe the two Priſms that are in the Pyramid A B C G are equal to one another, as alſo thoſe two that are in the Pyramid D E F H; it ſhall be, as the Priſm whoſe Baſe is the Parallelogram K L X B, and the oppoſite Baſe to that the Right Line M O, is to the Priſm whoſe Baſe is the Triangle L X C, and the oppoſite Baſe to that the Triangle O M N, ſo is the Priſm whoſe Baſe is the Parallelogram E P R Q, and the oppoſite Baſe to that the Right Line S T, to the Priſm whoſe Baſe is the Triangle R Q F, and the oppoſite Baſe to that the Triangle S T Y: Therefore (by compounding), as the Priſms K B X L M O, L X C M N O, together, are to the Priſm L X C M N O, ſo the Priſms P E Q R S T, R Q F S T Y, together, are to the Priſm R Q F S T Y: And (by Alternation), as the Priſms K B X L M O, L X C M N O, together, are to the Priſms P E Q R S T, R Q F S T Y, together, ſo is the Priſm L X C M N O to the Priſm R Q F S T Y: But as the Priſm L X C M N O is to the Priſm R Q F S T Y, ſo has the Baſe L X C been proved to be to the Baſe R F Q; and ſo the Baſe A B C to the Baſe D E F: Therefore, alſo, as the Triangle A B C is to the Triangle D E F, ſo are the two Priſms that are in the Pyramid A B C G, to the two Priſms that are in the Pyramid D E F H. If, in the ſame manner, each of the Pyramids O M N G, S T Y H, made by the former Diviſion, be divided, it ſhall be, as the Baſe O M N is to the Baſe S T Y, ſo the two Priſms that are in the Pyramid O M N G, to the two Priſms that are in the Pyramid S T Y H. But as the Baſe O M N is to the Baſe S T Y, ſo is the Baſe A B C to the Baſe D E F: Therefore, as the Baſe A B C is to the Baſe D E F, ſo are the two Priſms that are in the Pyramid A B C G, to the two Priſms that are in the Pyramid D E F H; and ſo the two Priſms that are in the Pyramid O M N G, to the two Priſms that are in the Pyramid S T Y H; and ſo the four to the four. We demon-

* 11. 5.

demonſtrate the ſame of Priſms made by the Diviſion of the Pyramids A K L O, D P R S; and, *of all other Priſms, being equal in Multitude*; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

Pyramids of the ſame Altitude, and having triangular Baſes, are to one another as their Baſes.

LET there be two Pyramids of the ſame Altitude, having the triangular Baſes A B C, D E F, whoſe Vertices are the Points G, H, I ſay, as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to the Pyramid D E F H.

For, if it be not ſo, then it ſhall be as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to ſome Solid, greater or leſs than the Pyramid D E F H. Firſt, let it be to a Solid leſs, which let be Z; and divide the Pyramid D E F H into two Pyramids equal to each other, and ſimilar to the Whole, and into two equal Priſms; then theſe two Priſms are greater than the Half of the whole Pyramid: And, again, let the Pyramids, made by the former Diviſion, be divided after the ſame manner; and let this be done continually, until the Pyramids in the Pyramid D E F H are leſs than the Exceſs by which the Pyramid D E F H, exceeds the Solid Z. Let theſe, for Example, be the Pyramids D P R S, S T Y H; then the Priſms remaining in the Pyramid D E F H, are greater than the Solid Z: Alſo, let the Pyramid A B C G be divided into the ſame Number of ſimilar Parts as the Pyramid D E F H is; and then, as the Baſe A B C is to the Baſe D E F, ſo * the Priſms that are in the Pyramid A B C G, to the Priſms that are in the Pyramid D E F H. But as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to the Solid Z ‡; and therefore, as the Pyramid A B C G is to the Solid Z, ſo are the Priſms that are in the Pyramid A B C G, to the Priſms that are in the Pyramid D E F H. But the the Pyramid A B C G is greater than the Priſms that are in it; wherefore, alſo, the Solid Z is greater than the

* 4 *of this.*

‡ *By Hyp.*

the Priſms that are in the Pyramid D E F H. But it is leſs * alſo; which is abſurd; therefore the Baſe A B C to the Baſe D E F, is not as the Pyramid A B C G to ſome Solid leſs than the Pyramid D E F H. After the ſame manner we demonſtrate, that the Baſe D E F to the Baſe A B C, is not as the Pyramid D E F H to ſome Solid leſs than the Pyramid A B C G: Therefore, I ſay, neither is the Baſe A B C to the Baſe D E F, as the Pyramid A B C G to ſome Solid greater than the Pyramid D E F H. For, if this be poſſible, let it be to the Solid I, greater than the Pyramid D E F H; then (by Inverſion) the Baſe D E F ſhall be to the Baſe A B C, as the Solid I to the Pyramid A B C G: But ſince the Solid I is greater than the Pyramid E D F H, it ſhall be, as the Solid I is to the Pyramid A B C G, ſo is the Pyramid D E F H to ſome Solid leſs than the Pyramid A B C G; and ſo, as the Baſe D E F is to the Baſe A B C, ſo is the Pyramid D E F H to ſome Solid leſs than the Pyramid A B C G, which is abſurd, as juſt now has been proved: Therefore the Baſe A B C to the Baſe D E F, is not as the Pyramid A B C G to ſome Solid greater than the Pyramid D E F H. But it has been alſo proved, that the Baſe A B C to the Baſe D E F, is not as the Pyramid A B C G to ſome Solid leſs than the Pyramid D E F H; wherefore, as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to the Pyramid D E F H. Therefore, *Pyramids of the ſame Altitude, and having triangular Baſes, are to one another as their Baſes*; which was to be demonſtrated.

* *From what has been before proved.*

PROPOSITION VI.

Theorem.

Pyramids of the ſame Altitude, and having polygonous Baſes, are to one another as their Baſes.

LET there be Pyramids of the ſame Altitude, which have the polygonous Baſes A B C D E, F G H K L; and let their Vertices be the Points M, N. I ſay, as the Baſe A B C D E is to the Baſe F G H K L, ſo is the Pyramid A B C D E M to the Pyramid F G H K L N.

For, let the Baſe ABCDE be divided into the Triangles ABC, ACD, ADE; and the Baſe FGHKL into the Triangles FGH, FHK, FKL; and let the Pyramids be conceived upon every one of thoſe Triangles, of the ſame Altitude with the Pyramids ABCDEM, FGHKLN: Then, becauſe the Triangle ABC is to the Triangle ACD, as * the Pyramid ABCM is to the Pyramid ACDM; and (by compounding) as the Trapezium ABCD is to the Triangle ACD, ſo is the Pyramid ABCDM to the Pyramid ACDM: But as the Triangle ACD is to the Triangle ADE, ſo is * the Pyramid ACDM to the Pyramid ADEM. Wherefore (by Equality of Proportion), as the Baſe ABCD is to the Baſe ADE, ſo is the Pyramid ABCDM to the Pyramid ADEM: And again (by Compoſition of Proportion), as the Baſe ABCDE is to the Baſe ADE, ſo is the Pyramid ABCDEM to the Pyramid ADEM. For the ſame Reaſon, as the Baſe FGHKL is to the Baſe FKL, ſo is the Pyramid FGHKLN to the Pyramid FKLN: And ſince there are two Pyramids ADEM, FKLN, having triangular Baſes, and the ſame Altitude; the Baſe ADE ſhall be * to the Baſe FKL, as the Pyramid ADEM to the Pyramid FKLN: And ſince the Baſe ABCDE is to the Baſe ADE, as the Pyramid ABCDEM is to the Pyramid ADEM; and as the Baſe ADE is to the Baſe FKL, ſo is the Pyramid ADEM to the Pyramid FKLN; it ſhall be (by Equality of Proportion), as the Baſe ABCDE is to the Baſe FKL, ſo is the Pyramid ABCDEM to the Pyramid FKLN: But as the Baſe FKL is to the Baſe FGHKL, ſo was the Pyramid FKLN to the Pyramid FGHKLN. Wherefore, again, (by Equality of Proportion), as the Baſe ABCDE is to the Baſe FGHKL, ſo is the Pyramid ABCDEM to the Pyramid FGHKLN. Therefore, *Pyramids of the ſame Altitude, and having polygonous Baſes, are to one another as their Baſes*; which was to be demonſtrated.

* 5 *of this.*

If the Baſes had not conſiſted of equal Numbers of Sides, the Demonſtration had been the ſame.

PROPOSITION VII.

Theorem.

Every Priſm, having a triangular Baſe, may be divided into three Pyramids, equal to one another, and having triangular Baſes.

LET there be a Priſm whoſe Baſe is the Triangle ABC, and the oppoſite Baſe to that the Triangle DEF. I ſay, the Priſm ABCDEF may be divided into three equal Pyramids, that have triangular Baſes.

For, join BD, EC, CD: Then, becauſe ABED is a Parallelogram, whoſe Diameter is BD, the Triangle ABD ſhall be * equal to the Triangle EBD. Therefore * 34. 1. the Pyramid whoſe Baſe is the Triangle ABD, and Vertex the Point C, is † equal to the Pyramid † 6 of this. whoſe Baſe is the Triangle EDB, and Vertex the Point C. But the Pyramid, whoſe Baſe is the Triangle EDB, and Vertex the Point C, is the ſame as the Pyramid whoſe Baſe is the Triangle EBC, and Vertex the Point D; for they are contained under the ſame Planes: Therefore the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C, is equal to the Pyramid whoſe Baſe is the Triangle EBC, and Vertex the Point D. Again, becauſe FCBE is a Parallelogram, whoſe Diameter is CE, the Triangle ECF ſhall be * equal to the Triangle CBE; and ſo the Pyramid whoſe Baſe is the Triangle BEC, and Vertex the Point D, is † equal to the Pyramid whoſe Baſe is the Triangle ECF, and Vertex the Point D. But the Pyramid, whoſe Baſe is the Triangle BCE, and Vertex the Point D, has been proved equal to the Pyramid whoſe Baſe is the Triangle ABD, and Vertex the Point C: Wherefore, alſo, the Pyramid, whoſe Baſe is the Triangle CEF, and Vertex the Point D, is equal to the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C: Therefore, *the Priſm* ABCDEF, *is divided into three Pyramids equal to one another, and having triangular Baſes.* And becauſe the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C, is the ſame with the Pyramid whoſe Baſe is the Triangle CAB, and Vertex the Point D; for they are contained under the ſame Planes; and the Pyramid, whoſe Baſe is the Triangle A B D, and Vertex the Point C, has been proved to be a third Part of the Priſm, whoſe Baſe is the Triangle A B C, and the oppoſite Baſe to that the the Triangle D E F: Therefore, alſo, *the Pyramid, whoſe Baſe is the Triangle* A B C, *and Vertex the Point* D, *is a third Part of the Priſm, having the ſame Baſe,* viz. *the Triangle* A B C, *and the oppoſite Baſe the Triangle* D E F; which was to be demonſtrated.

Coroll. 1. It is manifeſt from hence, that every Pyramid is a third Part of a Priſm, having the ſame Baſe, and an equal Altitude; becauſe, if the Baſe of a Priſm, as alſo the oppoſite Baſe, be of any other Figure, it may be divided into Priſms having triangular Baſes.

2. Priſms of the ſame Altitude are to one another as their Baſes.

PROPOSITION VIII.

THEOREM.

Similar Pyramids, having triangular Baſes, are in a triplicate Proportion of their homologous Sides.

LET there be two Pyramids ſimilar and alike ſituate, having the triangular Baſes A B C, D E F; and let their Vertices be the Points G, H. I ſay the Pyramid A B C G to the Pyramid D E F H, has a Proportion triplicate of that which B C has to E F.

For, compleat the ſolid Parallelepipedon B G M L, E H P O; then, becauſe the Pyramid A B C G is ſimilar to the Pyramid D E F H, the Angle A B C ſhall
* *Def.* 9. 11. be * equal to the Angle D E F, the Angle G B C equal to the Angle H E F, and the Angle A B G equal to the Angle D E H: And A B is to D E, as B C is to E F; and ſo is B G to E H. Therefore becauſe the Angle A B C is equal to the Angle D E F; and the Sides about the equal Angles are pro-
† 6. 6. portional; the Parallelogram B M ſhall be † ſimilar to the Parallelogram E P. For the ſame Reaſon, the Pa-

rallelogram B N is ſimilar to the Parallelogram E R, and the Parallelogram B K to the Parallelogram E X. Therefore three Parallelograms B M, B K, B N, are ſimilar to three Parallelograms E P, E X, E R. But the three Parallelograms B M, B K, B N, are equal and ſimilar to the three oppoſite ones; as alſo the three Parallelograms E P, E X, E R: Therefore the Solids B G M L, E H P O, are contained under equal Numbers of ſimilar and equal Planes; and, conſequently, the Solid B G M L is ſimilar to the Solid E H P O. But ſimilar ſolid Parallelepipedons are * to each other in a triplicate Proportion of their homologous Sides; therefore the Solid B G M L to the Solid E H P O, has a Proportion triplicate of that which the homologous Side B C has to the homologous Side E F. But as the Solid B G M L is to the Solid E H P O, ſo is † the Pyramid A B C G to the Pyramid D E F H; for the Pyramid is the one ſixth Part of that Solid, ſince the Priſm, which is the Half of the ſolid Parallelepipedon, is triple of the Pyramid. Wherefore, the *Pyramid*, A B C G *to the Pyramid* D E F H, *ſhall have a triplicate Proportion to that which* B C *has to* E F; which was to be demonſtrated.

* 33. 11.

† 15. 5.

Coroll. From hence it is manifeſt, that ſimilar Pyramids having polygonous Baſes, are to one another in a triplicate Proportion of their homologous Sides. For, if they be divided into Pyramids having triangular Baſes; becauſe their ſimilar polygonous Baſes are divided into ſimilar Triangles equal in Number, and homologous to the Wholes; it ſhall be, as one Pyramid, having a triangular Baſe in one of the Pyramids, is to a Pyramid having a triangular Baſe in the other Pyramid; ſo are all the Pyramids, having triangular Baſes in one Pyramid, to all the Pyramids having triangular Baſes in the other Pyramid; that is, ſo is one of the Pyramids, having the polygonous Baſe, to the other: But a Pyramid, having a triangular Baſe, to a Pyramid having a triangular Baſe, is in a triplicate Proportion of the homologous Sides. Therefore one Pyramid, having a polygonous Baſe, to another Pyramid having a ſimilar Baſe, is in a triplicate Proportion of their homologous Sides.

PRO-

PROPOSITION IX.

Theorem.

The Baſes and Altitudes of equal Pyramids, having triangular Baſes, are reciprocally proportional; and thoſe Pyramids, having triangular Baſes, whoſe Baſes and Altitudes are reciprocally proportional, are equal.

LET there be equal Pyramids, having the triangular Baſes A B C, D E F, and Vertices the Points G, H. I ſay, the Baſes and Altitudes of the Pyramids A B C G, D E F H, are reciprocally proportional; that is, as the Baſe A B C is to the Baſe D E F, ſo is the Altitude of the Pyramid D E F H to the Altitude of the Pyramid A B C G.

For, compleat the ſolid Parallelepipedons B G M L, E H P O; then, becauſe the Pyramid A B C G is equal to the Pyramid D E F H; and the Solid B G M L is ſextuple the Pyramid A B C G; and the Solid E H P O ſextuple the Pyramid D E F H; the Solid B G M L ſhall be * equal to the Solid E H P O. But the Baſes and Altitudes of equal ſolid Parallelepipedons are reciprocally proportional; therefore, as the Baſe B M is to
† 34. 11. the Baſe E P, ſo is † the Altitude of the Solid E H P O to the Altitude of the Solid B G M L. But as the Baſe
* 15. 5. B M is to the Baſe E P, ſo is * the Triangle A B C to the Triangle D E F; therefore, as the Triangle A B C is to the Triangle D E F, ſo is the Altitude of the Solid E H P O to the Altitude of the Solid B G M L. But the Altitude of the Solid E H P O is the ſame as the Altitude of the Pyramid D E F H; and the Altitude of the Solid B G M L, the ſame as the Altitude of the Pyramid A B C G; therefore, as the Baſe A B C is to the Baſe D E F, ſo is the Altitude of the Pyramid D E F H to the Altitude of the Pyramid A B C G: Wherefore the Baſes and Altitudes of the equal Pyramids A B C G, D E F H, are reciprocally proportional. And if the Baſes and Altitudes of the Pyramids A B C G, D E F H, are reciprocally proportional; that is, if the Baſe A B C to the Baſe D E F,

be as the Altitude of the Pyramid D E F H to the Altitude of the Pyramid A B C G; I ſay, the Pyramid A B C G is equal to the Pyramid D E F H: For, the ſame Conſtruction remaining, becauſe the Baſe A B C to the Baſe D E F, is as the Altitude of the Pyramid D E F H to the Altitude of the Pyramid A B C G; and as the Baſe A B C is to the Baſe D E F, ſo is the Parallelogram B M to the Parallelogram E P; therefore the Parallelogram B M to the Parallelogram E P, ſhall be alſo as the Altitude of the Pyramid D E F H is to the Altitude of the Pyramid A B C G. But as the Altitude of the Pyramid D E F H is the ſame as the Altitude of the ſolid Parallelepipedon E H P O, and the Altitude of the Pyramid A B C G, the ſame as the Altitude of the ſolid Parallelepipedon B G M L; therefore the Baſe B M to the Baſe E P, will be as the Altitude of the ſolid Parallelepipedon E H P O to the Altitude of the ſolid Parallelepipedon B G M L. But thoſe ſolid Parallelepipedons, whoſe Baſes and Altitudes are reciprocally proportional, are † equal to each other; therefore the ſolid † 34. 11. Parallelepipedon B G M L, is equal to the ſolid Parallelepipedon E H P O: Now the Pyramid A B C G is a ſixth Part of the Solid B G M L; and, in like manner, the Pyramid D E F H is a ſixth Part of the Solid E H P O; therefore the Pyramid A B C G is equal to the Pyramid D E F H. Wherefore, *the Baſes and Altitudes of equal Pyramids, having triangular Baſes, are reciprocally proportional; and thoſe Pyramids, having triangular Baſes, whoſe Baſes and Altitudes are reciprocally proportional, are equal*; which was to be demonſtrated.

PROPOSITION X.

THEOREM.

Every Cone is a third Part of a Cylinder, having the ſame Baſe, and an equal Altitude.

LET a Cone have the ſame Baſe as a Cylinder; *viz.* the Circle A B C D, and an Altitude equal to it. I ſay, the Cone is a third Part of the Cylinder; that is, the Cylinder is triple to the Cone.

For, if the Cylinder be not triple to the Cone, it ſhall be greater or leſs than triple thereof. Firſt, let it be greater than triple to the Cone, and let the Square A B C D be deſcribed in the Circle A B C D; then the Square A B C D, is greater than one half of the Circle A B C D. Now let a Priſm be erected upon the Square A B C D, having the ſame Altitude as the Cylinder, and this Priſm will be greater than one half of the Cylinder; becauſe, if a Square be circumſcribed about the Circle A B C D, the inſcribed Square will be one half of the circumſcribed Square; and if a Priſm be erected upon the circumſcribed Square of the ſame Altitude as the Cylinder, ſince Priſms are to * one another as their Baſes, the Priſm erected upon the Square A B C D, is one half of the Priſm erected upon the Square deſcribed about the Circle A B C D. But the Cylinder is leſs than the Priſm erected on the Square deſcribed about the Circle A B C D; therefore the Priſm erected on the Square A B C D, having the ſame Height as the Cylinder, is greater than one half of the Cylinder. Let the Circumferences A B, B C, C D, D A, be biſected in the Points E, F, G, H; and join A E, E B, B F, F C, C G, G D, D H, H A: Then each of the Triangles A E B, B F C, C G D, D H A, is † greater than the half of each of the Segments in which they ſtand. Let Priſms be erected from each of the Triangles A E B, B F C, C G D, D H A, of the ſame Altitude as the Cylinder; then every one of theſe Priſms erected is greater than half its correſpondent Segment of the Cylinder. For, becauſe, if Parallels be drawn thro' the Points E, F, G, H, to A B, B C, C D, D A, and Parallelograms be compleated on the ſaid A B, B C, C D, D A, on which are erected ſolid Parallelepipedons of the ſame Altitude as the Cylinder; then each of thoſe Priſms that are on the Triangles A E B, B F C, C G D, D H A, are Halves ‡ of each of the ſolid Parallelepipedons; and the Segments of the Cylinder are leſs than the erected ſolid Parallelepipedons; and, conſequently, the Priſms that are on the Triangles A E B, B F C, C G D, D H A, are greater than the Halves of the Segments of the Cylinder: And ſo, biſecting the other Circumferences, joining Right Lines, and on every one of the Triangles erecting Priſms of the ſame Height as the

* 2 *Cor.* 7. *of this.*

† *This follows from* 2 *of this.*

‡ 32. 11.

the Cylinder, and doing this continually, we ſhall at laſt have certain Portions of the Cylinder left, that are leſs than the Exceſs by which the Cylinder exceeds triple the Cone.

Now, let theſe Portions remaining be A E, E B, B F, F C, C G, G D, D H, H A; then the Priſm remaining, whoſe Baſe is the Polygon A E B F C G D H, and Altitude equal to that of the Cylinder's is greater than the Triple of the Cone. But the Priſm, whoſe Baſe is the Polygon A E B F C G D H, and Altitude the ſame as that of the Cylinder's, is * triple of the Pyramid, whoſe Baſe is the Polygon A E B F C G D H, and Vertex the ſame as that of the Cone; and therefore the Pyramid, whoſe Baſe is the Polygon A E B F C G D H, and Vertex the ſame as that of the Cone, is greater than the Cone, whoſe Baſe is the the Circle A B C D: But it is leſs alſo (for it is comprehended by it) which is abſurd; therefore the Cylinder is not greater than triple the Cone. I ſay, it is neither leſs than triple the Cone: For, if it be poſſible, let the Cylinder be leſs than triple the Cone; then (by Inverſion) the Cone ſhall be greater than a third Part of the Cylinder: Let the Square A B C D be deſcribed in the Circle A B C D; then the Square A B C D is greater than half of the Circle A B C D: And let a Pyramid be erected on the Square A B C D, having the ſame Vertex as the Cone; then the Pyramid erected is greater than one half of the Cone; becauſe, as has been already demonſtrated, if a Square be deſcribed about the Circle, the Square A B C D ſhall be half thereof: And if ſolid Parallelepipedons be erected upon the Squares of the ſame Altitude as the Cone, which are alſo Priſms; then the Priſm erected on the Square A B C D is one half of that erected on the Square deſcribed about the Circle; for they are to each other as their Baſes, and ſo likewiſe are their third Parts: Therefore the Pyramid, whoſe Baſe is the Square A B C D, is one half of that Pyramid erected upon the Square deſcribed about the Circle. But the Pyramid erected upon the Square deſcribed about the Circle is greater than the Cone, for it comprehends it; therefore the Pyramid, whoſe Baſe is the Square A B C D, and Vertex the ſame as that of the Cone,

* 1 Cor. 7. of this.

is greater than one half of the Cone. Biſect the Circumferences A B, B C, C D, D A, in the Points E, F, G, H; and join A E, E B, B F, F C, C G, G D, D H, H A; and then each of the Triangles A E B, B F C, C G D, D H A, is greater than one half of each of the Segments they are in. Let Pyramids be erected upon each of the Triangles A E B, B F C, C G D, D H A, having the ſame Vertex as the Cone; then each of theſe Pyramids, thus erected, is greater than one half of the Segment of the Cone in which it is; and ſo, biſecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon every of the Triangles having the ſame Altitude as the Cone, and doing this continually, we ſhall at laſt have Segments of the Cone left, that will be leſs than the Exceſs by which the Cone exceeds the one third Part of the Cylinder: Let theſe Segments be thoſe that are on A E, E B, B F, F C, C G, G D, D H, H A; and then the remaining Pyramid, whoſe Baſe is the Polygon A E B F C G D H, and Vertex the ſame as that of the Cone, is greater than a third Part of the Cylinder: But the Pyramid, whoſe Baſe is the Polygon A E B F C G D H, and Vertex the ſame as that of the Cone, is one third Part of the Priſm whoſe Baſe is the Polygon A E B F C G D H, and Altitude the ſame as that of the Cylinder: Therefore the Priſm, whoſe Baſe is the Polygon A E B F C G D H, and Altitude the ſame as that of the Cylinder, is greater than the Cylinder, whoſe Baſe is the Circle A B C D; but it is leſs alſo (as being comprehended thereby); which is abſurd; therefore the Cylinder is not leſs than triple of the Cone: But it has been proved alſo not to be greater than triple of the Cone; therefore the Cylinder is neceſſarily triple of the Cone. Wherefore, *every Cone is a third Part of a Cylinder, having the ſame Baſe, and an equal Altitude*; which was to be demonſtrated.

PRO-

PROPOSITION XI.

THEOREM.

Cones and Cylinders, of the ſame Altitude, are to one another as their Baſes.

LET there be Cones and Cylinders of the ſame Altitude, whoſe Baſes are the Circles A B C D, E F G H, Axes K L, M N, and Diameters of the Baſes A C, E G. I ſay, as the Circle A B C D is to the Circle E F G H, ſo is the Cone A L to the Cone E N.

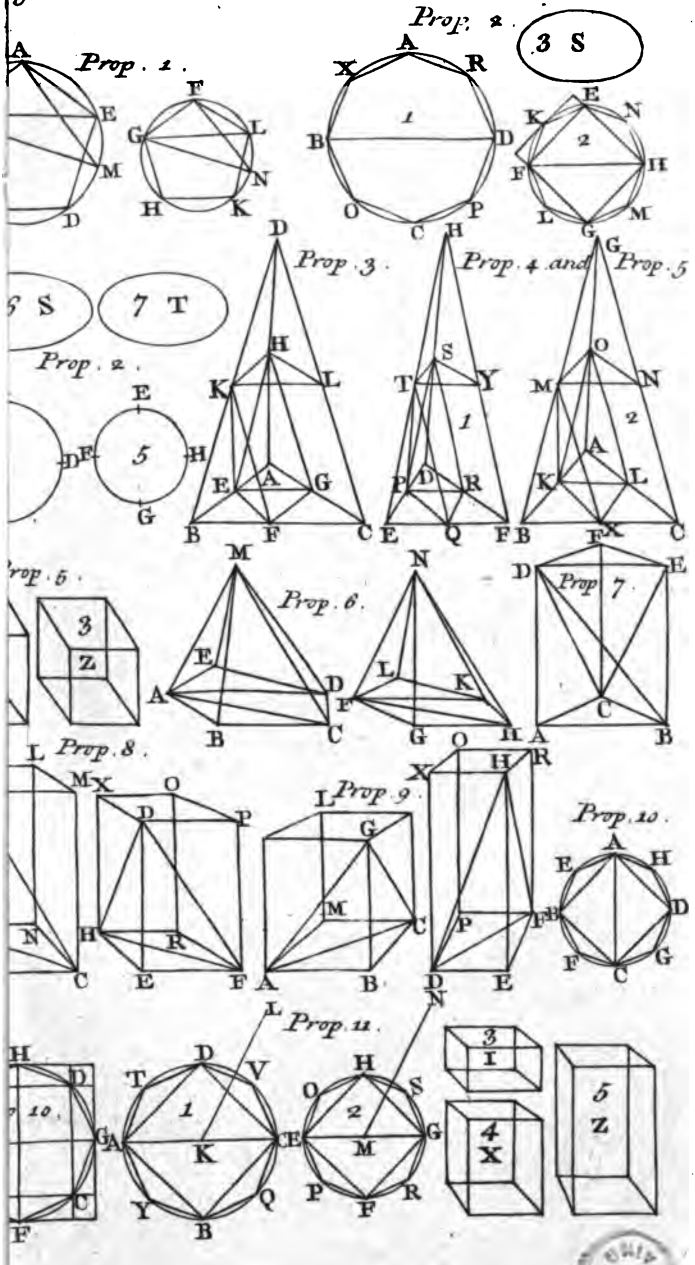
For, if it be not ſo, it ſhall be, as the Circle A B C D is to the Circle E F G H, ſo is the Cone A L to ſome Solid either leſs or greater than the Cone E N. Firſt, let it be to the Solid X leſs than the Cone; and let the Solid I be equal to the Exceſs of the Cone E N above the Solid X: Then the Cone E N is equal to the Solids X and I. Let the Square E F G H be deſcribed in the Circle E F G H, which Square is greater than one half of the Circle, and erect a Pyramid upon the Square E F G H, of the ſame Altitude as the Cone; therefore the Pyramid erected is greater than one half of the Cone. For if we deſcribe a Square about the Circle, and a Pyramid be erected thereon, of the ſame Altitude as the Cone; the Pyramid inſcribed will be one half of the Pyramid circumſcribed; for they are * to one another as their Baſes; and the Cone is leſs than the circumſcribed Pyramid: Therefore the Pyramid, whoſe Baſe is the Square E F G H, and Vertex the ſame as that of the Cone, is greater than one half of the Cone. Biſect the Circumferences E F, F G, G H, H E, in the Points P, R, S, O; and join H O, O E, E P, P F, F R, R G, G S, S H; then each of the Triangles H O E, E P F, F R G, G S H, is greater than one half of the Segment of the Circle wherein it is. Let a Pyramid be raiſed upon every one of the Triangles H O E, E P F, F R G, G S H, of the ſame Altitude as the Cone; then each of thoſe erected Pyramids is greater than one half of its correſpondent Segment of the Cone: And ſo biſecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon each of the Triangles, of the ſame Altitude

* 6 of this.

as that of the Cone; and doing this continually, there will at laſt be left Segments of the Cone that will together be leſs than the Solid I. Let thoſe be the Segments that are on HO, OE, EP, PF, FR, RG, GS, SH; therefore the Pyramid remaining, whoſe Baſe is the Polygon HOEPFRGS, and Altitude the ſame as that of the Cone, is greater than the Solid X. Let the Polygon D T A Y B Q C V be deſcribed in the Circle A B C D, ſimilar and alike ſituate to the Polygon H O E P F R G S; and let a Pyramid be erected thereon of the ſame Altitude as the Cone A L: Then,
* 1 *of this.* becauſe the Square of A C to the Square of E G, is * as the Polygon D T A Y B Q C V to the Polygon
† 2 *of this.* H O E P F R G S; and the Square of A C is † to the Square of E G, as the Circle A B C D is to the Circle E F G H; it ſhall be, as the Circle A B C D is to the Circle E F G H, ſo is the Polygon D T A Y B Q C V to the Polygon H O E P F R G S. But as the Circle A B C D is to the Circle E F G H, ſo is the Cone A L to the Solid X (by *Hyp.*): And as the Polygon D T A Y B Q C V is to the Polygon H O E P F R G S,
‡ 6 *of this.* ſo is ‡ the Pyramid, whoſe Baſe is the Polygon D T A Y B Q C V, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon H O E P F R G S, and Vertex the Point N. Therefore, as the Cone A L is to the Solid X, ſo is the Pyramid, whoſe Baſe is the Polygon D T A Y B Q C V, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon H O E P F R G S, and Vertex the Point N. But the Cone A L is greater than the Pyramid that is in it; therefore the Solid X is greater than the Pyramid that is in the Cone E N; but it was put leſs, which is abſurd. Therefore the Circle A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid leſs than the Cone E N. In like manner it is demonſtrated, that the Circle E F G H to the Circle A B C D, is not as the Cone E N to ſome Solid leſs than the Cone A L: I ſay, moreover, that the Circle A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid greater than the Cone E N. For, if it be poſſible, let it be to the Solid Z greater than the Cone; then (by Inverſion), as the Circle E F G H is to the Circle A B C D, ſo ſhall the Solid Z be to the Cone A L. But ſince the Solid Z is greater than the

Cone

g.260. Book 12. Plate 1.



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Cone E N, it ſhall be, as the Solid Z is to the Cone A L, ſo is the Cone E N to ſome Solid leſs than the Cone A L; and therefore, as the Circle E F G H is to the Circle A B C D, ſo is the Cone E N to ſome Solid leſs than the Cone A L; which has been proved to be impoſſible. Therefore the Solid A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid greater than the Cone E N. It has alſo been proved, that the Circle A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid leſs than the Cone E N; therefore, as the Circle A B C D is to the Circle E E G H, ſo is the Cone A L to the Cone E N: But as Cone is to Cone, ſo is * Cylinder * 15. 5.
to Cylinder; for each Cylinder is triple of each Cone; and therefore, as the Circle A B C D is to the Circle E F G H, ſo are Cylinders and Cones ſtanding on them, of the ſame Altitude. Wherefore, *Cones and Cylinders of the ſame Altitude, are to one another as their Baſes*; which was to be demonſtrated.

PROPOSITION XII.

THEOREM.

Similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes.

LET there be ſimilar Cones and Cylinders, whoſe Baſes are the Circles A B C D, E F G H, and Diameters of the Baſes B D, F H, and Axes of the Cones or Cylinders K L, M N. I ſay, the Cone, whoſe Baſe is the Circle A B C D, and Vertex the Point L, to the Cone whoſe Baſe is the Circle E F G H, and Vertex the Point N, hath a triplicate Proportion of that which B D has to F H.

For, if the Cone A B C D L to the Cone E F G H N, has not a triplicate Proportion of that which B D has to F H; the Cone A B C D L ſhall have that triplicate Proportion to ſome Solid, either leſs or greater than the Cone E F G H N. Firſt, let it have that triplicate Proportion to the Solid X, leſs than the Cone E F G H N; and let the Square

EFGH be deſcribed in the Circle EFGH, which will be greater than one half of the Circle EFGH; and erect a Pyramid on the Square EFGH, of the ſame Altitude with the Cone, then that Pyramid is greater than one half of the Cone. And ſo let the Circumferences EF, FG, GH, HE, be biſected in the Points O, P, R, S; and join EO, OF, FP, PG, GR, RH, HS, SE; then each of the Triangles EOF, FPG, GRH, HSE, is greater than one half of the Segment of the Circle EFGH, in which it is; and erect a Pyramid upon each of the the Triangles EOF, FPG, GRH, HSE, having the ſame Altitude as the Cone: Then each of the Pyramids, thus erected, is greater than half its correſponding Segment of the Cone; wherefore, biſecting the remaining Circumferences, joining Right Lines, and erecting Pyramids upon each of the Triangles, having the ſame Vertex as the Cone; and doing this continually, we ſhall leave, at laſt, certain Segments of the Cone, that ſhall be leſs than the Exceſs by which the Cone EFGHN exceeds the Solid X. Let theſe be the Segments that ſtand on EO, OF, FP, PG, GR, RH, HS, SE; then the remaining Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, is greater than the Solid X: Alſo, let the Polygon ATBYCVDQ be deſcribed in the Circle ABCD, ſimilar and alike ſituate to the Polygon EOFPGRHS; upon which erect a Pyramid having the ſame Altitude as the Cone; and let LBT be one of the Triangles containing the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L; as likewiſe NFO one of the Triangles containing the Pyramid EOFPGRHS, and Vertex the Point N; and let KT, MO, be joined: Then, becauſe the Cone ABCDL is ſimilar to the Cone EFGHN, it ſhall be, as BD is to FH, ſo is the Axis KL to the Axis MN: But as
* 15. 5. BD is to FH, ſo is * BK to FM; conſequently, as BK is to FM, ſo is KL to MN; and (by Alternation) as BK is to KL, ſo is FM to MN. And ſince each is perpendicular, and the Sides about the equal Angles BKL, FMN are proportional; the
† 6. 6. Triangle BKL ſhall be † ſimilar to the Triangle FMN. Again, becauſe BK is to KT, as FM is

to MO, the Sides are proportional about the equal Angles BKT, FMO, (for the Angle BKT is the ſame Part of the four Right Angles at the Centre K, as the Angle FMO is of the four Right Angles at the Centre M); therefore the Triangle BKT ſhall be * ſimilar to the Triangle FMO. And becauſe it has been proved, that BK is to KL, as FM is to MN; and BK is equal to KT; and FM to MO; it ſhall be, as TK is to KL, ſo is OM to MN; and the proportional Sides are about the equal Angles TKL, OMN; for they are Right Angles: Therefore the Triangle LKT ſhall be ſimilar to the Triangle MNO. And ſince, by the Similarity of the Triangles BKL, FMN, it is, as LB is to BK, ſo is NF to FM; and, by the Similarity of the Triangles BKT, FMO, it is, as KB is to BT, ſo is MF to FO; it ſhall be (by Equality of Proportion), as LB is to BT, ſo is NF to FO. Again, ſince, by the Similarity of the Triangles LTK, NOM, it is, as LT is to TK, ſo is NO to OM; and, by the Similarity of the Triangles KBT, OMF, it is, as KT is to TB, ſo is MO to OF; it ſhall be (by Equality of Proportion), as LT is to TB, ſo is NO to OF. But it has been proved, that TB is to BL, as OF is to FN; wherefore, again (by Equality of Proportion), as TL is to LB, ſo is ON to NF; and therefore the Sides of the Triangles LTB, NOF, are proportional; and ſo the Triangles LTB, NOF, are equiangular and ſimilar to each other; and, conſequently, the Pyramid, whoſe Baſe is the Triangle BKT, and Vertex the Point L, is ſimilar to the Pyramid whoſe Baſe is the Triangle FMO, and Vertex the Point N; for they are contained under ſimilar Planes equal in Multitude: But ſimilar Pyramids that have Triangular Baſes, are † to one another in the triplicate Proportion of their homologous Sides; therefore the Pyramid BKTL to the Pyramid FMON, has a triplicate Proportion of that which BK has to FM. In like manner, drawing Right Lines from the Points A, Q, D, V, C, Y, to K; as alſo others from the Points E, S, H, R, G, P, to M; and erecting Pyramids on the Triangles having the ſame Vertices as the Cones, we demonſtrate, that every Pyramid of one Cone, to every one of the other

* 6. 6.

† 8 *of this.*

Cone, has a triplicate Proportion of that which the Side B K has to the homologous Side M F, that is, which B D has to F H. But as one of the Antecedents is to one of the Conſequents, ſo are ‡ all the Antecedents to all the Conſequents. ‡ 12. 5. Therefore, as the Pyramid B K T L is to the Pyramid F M O N, ſo is the whole Pyramid, whoſe Baſe is the Polygon A T B Y C V D Q, and Vertex the Point L, to the whole Pyramid, whoſe Baſe is the Polygon E O F P G R H S, and Vertex the Point N. Wherefore the Pyramid, whoſe Baſe is the Polygon A T B Y C V D Q, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon E O F P G R H S, and Vertex the Point N, has a triplicate Proportion of that which B D hath to F H. But the Cone whoſe Baſe is the Circle A B C D, and Vertex the Point L, is ſuppoſed to have to the Solid X a triplicate Proportion of that which B D has to F H; therefore, as the Cone, whoſe Baſe is the Circle A B C D, and Vertex the Point L, is to the Solid X, ſo is the Pyramid whoſe Baſe is the Polygon A T B Y C V D Q, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon E O F P G R H S, and Vertex the Point N. But the ſaid Cone is greater than the Pyramid that is in it, for it comprehends it; therefore the Solid X alſo is greater than the Pyramid, whoſe Baſe is the Polygon E O F P G R H S, and Vertex the Point N; but it is alſo leſs, which is abſurd. Therefore the Cone, whoſe Baſe is the Circle A B C D, and Vertex the Point L, to ſome Solid leſs than the Cone, whoſe Baſe is the Circle E F G H, and Vertex the Point N, has not a triplicate Proportion of that which B D has to F H. In like manner we demonſtrate, that the Cone E F G H N, to ſome Solid leſs than the Cone A B C D L, has not a triplicate Proportion of that which F H has to B D. Laſtly, I ſay, the Cone A B C D L, to a Solid greater than the Cone E F G H N, has not a triplicate Proportion of that which B D has to F H: For, if this be poſſible, let it be ſo to ſome Solid Z greater than the Cone E F G H N; then (by Inverſion) the Solid Z, to the Cone A B C D L, has a triplicate Proportion of that which F H has to B D. But ſince the Solid Z is greater than the Cone E F G H N, the Solid Z ſhall be to the Cone ABCDL, as the Cone EFGHN is to ſome Solid leſs than the Cone ABCDL; and therefore the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, hath a triplicate Proportion of that which EA has to BD, which has been proved to be impoſſible; therefore the Cone ABCDL, to ſome Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH. It has been alſo demonſtrated, that the Cone ABCDL, to ſome Solid leſs than the Cone EFGHN, hath not a triplicate Proportion of that which BD has to FH; wherefore the Cone ABCDL, to the Cone EFGHN, has a triplicate Proportion of that which BD has to FH. But as Cone is to Cone, ſo is * Cylinder to Cylinder; * 15. 5. for a Cylinder, having the ſame Baſe as a Cone, and the ſame Altitude, is † triple of the Cone; ſince it is † 10 *of this.* demonſtrated, that every Cone is one third Part of a Cylinder, having the ſame Baſe, and equal Altitude: Therefore, alſo a Cylinder to a Cylinder has a triplicate Proportion of that which BD has to FH. Therefore, *ſimilar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes*; which was to be demonſtrated.

PROPOSITION XIII.

THEOREM.

If a Cylinder be divided by a Plane parallel to the oppoſite Planes; then, as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis.

LET the Cylinder AD be divided by the Plane GH, parallel to the oppoſite Planes AB, CD, and meeting the Axis EF in the Point K. I ſay, as the Cylinder BG is to the Cylinder GD, ſo is the Axis EK to the Axis KF.

For, let the Axis EF be both Ways produced to L and M; and put any Number of Lines EN, NL, *&c.* each equal to the Axis EK; and any Number of Lines FX, XM, *&c.* each equal to FK; and thro' the Points L, N, X, M, let Planes parallel to

A B, C D, paſs; and in thoſe Planes from L, N, X, M, as Centres, deſcribe the Circles O P, R S, T Y, V Q, each equal to A B, C D; and conceive the Cylinders P R, R B, D T, T Q, to be compleated: Then, becauſe the Axis L N, N E, E K, are equal to each other, the Cylinders P R, R B, B G, will be * to one another as their Baſes; and therefore the Cylinders P R, R B, B G, are equal: And ſince the Axis L N, N E, E K, are equal to each other; as alſo the Cylinders P R, R B, B G; and the Number of Lines L N, N E, E K, is equal to the Number of Cylinders P R, R B, B G; the Axis K L ſhall be the ſame Multiple of the Axis E K, as the Cylinder P G is of the Cylinder G B. For the ſame Reaſon, the Axis M K is the ſame Multiple of the Axis K F, as the Cylinder G Q is of the Cylinder G D. Now, if the Axis K L be equal to the Axis K M, the Cylinder P G ſhall be equal to the Cylinder G Q; if the Axis K L be greater than the Axis K M, the Cylinder P G ſhall be likewiſe greater than the Cylinder G Q; and, if leſs, leſs: Therefore, becauſe there are four Magnitudes, *viz.* the Axis E K, K F, and the Cylinders B G, G D; and there are taken their Equimultiples, namely, the Axis K L, and the Cylinder P G, the Equimultiples of the Axis E K, and the Cylinder B G; and the Axis K M, and the Cylinder G Q, the Equimultiples of the Axis K F, and the Cylinder G D: And it is demonſtrated, that if the Axis K L exceeds the Axis K M, the Cylinder P G will exceed the Cylinder G Q; and, if it be equal, equal; and, if leſs, leſs. Therefore, as the Axis E K is to the Axis K F, ſo † is the Cylinder B G to the Cylinder G D. Wherefore, *if a Cylinder be divided by a Plane parallel to the oppoſite Planes; then, as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis*; which was to be demonſtrated.

* 11 of this.

† Def. 5. 5.

PRO-

PROPOSITION XIV.

THEOREM.

Cones and Cylinders, being upon equal Baſes, are to one another as their Altitudes.

LET the Cylinders E B, F D, ſtand upon equal Baſes A B, C D. I ſay, as the Cylinder E B is to the Cylinder F D, ſo is the Axis G H to the Axis K L.

For, produce the Axis K L to the Point N; and put L N equal to the Axis G H; and let a Cylinder C M be conceived about the Axis L N: Then, becauſe the Cylinders E B, C M, have the ſame Altitude; they are * to one another as their Baſes. But * 11 *of this.* their Baſes are equal; therefore the Cylinders E B, C M, will be alſo equal. And becauſe the Cylinder F M is cut by a Plane C D, parallel to the oppoſite Planes, it ſhall be as the Cylinder C M is to the Cylinder F D, ſo is the Axis L N to the Axis K L. But the Cylinder C M is equal to the Cylinder E B; and the Axis L N to the Axis G H; therefore the Cylinder E B is to the Cylinder F D, as the Axis G H is to the Axis K L: And as the Cylinder E B is to the Cylinder F D, ſo is ‡ the Cone A B G to the Cone C D K; ‡ 15. 5. for the Cylinders are * triple of the Cones. There- * 10 *of this.* fore, as the Axis G H is to the Axis K L, ſo is the Cone A B G to the Cone C D K; and ſo the Cylinder E B to the Cylinder F D. Wherefore, *Cones and Cylinders, being upon equal Baſes, are to one another as their Altitudes*; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

The Baſes and Altitudes of equal Cones and Cylinders are reciprocally proportional; and Cones and Cylinders, whoſe Baſes and Altitudes are reciprocally proportional, are equal to one another.

LET the Baſes of the equal Cones and Cylinders be the Circles A B C D, E F G H, and their Diameters A C, E G; and Axis K L, M N; which are alſo the Altitudes of the Cones and Cylinders: And let the Cylinders A X, E O, be compleated. I ſay, the Baſes and Altitudes of the Cylinders A X, E O, are reciprocally proportional; that is, the Baſe A B C D is to the Baſe E F G H, as the Altitude M N is to the Altitude K L.

For, the Altitude K L is either equal to the Altitude M N, or not equal. Firſt, let it be equal; and the Cylinder A X is equal to the Cylinder E O. But Cylinders * 11 of this. and Cones, that have the ſame Altitude, are * to one another as their Baſes; therefore the Baſe A B C D is equal to the Baſe E F G H: And conſequently, as the Baſe A B C D is to the Baſe E F G H, ſo is the Altitude M N to the Altitude K L. But if the Altitude K L be not equal to the Altitude M N, let M N be the greater; and take P M, equal to L K, from M N; and let the Cylinder E O be cut thro' P by the Plane T Y S, parallel to the oppoſite Planes of the Circles E F G H, R O; and conceive E S to be a Cylinder, whoſe Baſe is the Circle E F G H, and Altitude P M: Then, becauſe the Cylinder A X is equal to the Cylinder E O, and E S is ſome other Cylinder; the Cylinder A X to the Cylinder E S, ſhall be as the Cylinder E O is to the Cylinder E S. But as the Cylinder * 11 of this. A X is to the Cylinder E S, ſo is * the Baſe A B C D to the Baſe E F G H; for the Cylinders A X, E S, have the ſame Altitude: And as the Cylinder † 11 of this. E O is to the Cylinder E S, ſo is † the Altitude M N to the Altitude M P; for the Cylinder E O is cut by the Plane T Y S, parallel to the oppoſite Planes. Therefore as the Baſe A B C D is to the Baſe E F G H, ſo is the Altitude M N to the Altitude M P. But the Altitude M P is equal to the Altitude K L; wherefore, as the Baſe A B C D is to the Baſe E F G H, ſo is the Altitude M N to the Altitude K L; and therefore, *the Baſes and Altitudes of the equal Cylinders* A X, E O, *are reciprocally proportional.*

And if the Baſes and Altitudes of the Cylinders A X, E O, are reciprocally proportional; that is, if the Baſe A B C D be to the Baſe E F G H, as the Altitude M N is to the Altitude K L; I ſay, the Cylinder A X is equal to the Cylinder E O. For, the ſame Conſtruction remaining, becauſe the Baſe A B C D is to the Baſe E F G H, as the Altitude M N is to the Altitude K L; and the Altitude K L is equal to the Altitude M P; it ſhall be, as the Baſe A B C D is to the Baſe E F G H, ſo is the Altitude M N to the Altitude M P. But as the Baſe A B C D is to the Baſe E F G H, ſo is the Cylinder A X to the Cylinder E S, for they have the ſame Altitude; alſo, as the Altitude M N is to the Altitude M P, ſo is ‡ the Cylinder E O ‡ 13 *of this.* to the Cylinder ES. Therefore, as the Cylinder A X is to the Cylinder E S, ſo is the Cylinder E O to the Cylinder E S: Wherefore, *the Cylinder* A X *is equal to the Cylinder* E O; which was to be demonſtrated.

In like manner we prove this in Cones.

PROPOSITION XVI.

Problem.

Two Circles being about the ſame Centre, to inſcribe in the greater a Polygon of equal Sides, even in Number, that ſhall not touch the leſſer Circle.

LET A B C D, E F G H, be two given Circles about the Centre K; it is required to inſcribe a Polygon of equal Sides, even in Number, in the Circle A B C D, not touching the leſſer Circle E F G H.

Draw the Right Line B D thro' the Centre K, as alſo A G, from the Point G, at Right Angles to B D, which produce to C; this Line will * touch the Cir- * 16. 3. cle E F G H: Then, biſecting the Circumference B A D, and again biſecting the Half thereof, and doing this continually, we ſhall have a Circumference left, at laſt, leſs than A D ‡. Let this Circumference ‡ *Lemma.* be L D, and draw L M, from the Point L, perpendicular to B D, which produce to N; and join L D, D N: And then L D is † equal to D N. And ſince † 29. 3. L N is parallel to A C, and A C touches the Circle E F G H, L N will not touch the Circle E F G H; and much leſs do the Right Lines L D, D N, touch the Circle. And if Right Lines, each equal to L D, be applied round the Circle A B C D, *we ſhall have a Polygon inſcribed therein of equal Sides, even in Number, that does not touch the leſſer Circle* E F G H; which was to be done.

PROPOSITION XVII.

Problem.

To deſcribe a ſolid Polyhedron, in the greater of two Spheres, having the ſame Centre, which ſhall not touch the Superficies of the leſſer Sphere.

LET two Spheres be ſuppoſed about the ſame Centre A; it is required to deſcribe a ſolid Polyhedron in the greater Sphere, not touching the Superficies of the leſſer Sphere.

Let the Spheres be cut by ſome Plane paſſing thro' the Centre; then the Sections will be Circles: For, * Def. 14. 11. becauſe a Sphere is * made by the turning of a Semicircle about the Diameter, which is at Reſt; in whatſoever Poſition the Semicircle is conceived to be, the Plane in which it is ſhall make a Circle in the Superficies of the Sphere. It is alſo manifeſt, that this Circle is a greater Circle, ſince the Diameter of the Sphere, which is likewiſe the Diameter of the Semicircle, is † 15. 3. † greater than all Right Lines that are drawn in the Circle or Sphere. Now, let B C D E be that Circle of the greater Sphere, and F G H of the leſſer Sphere; and let B D, C E, be two of their Diameters drawn at Right Angles to one another; let B D meet the leſſer Circle in the Point G, and let G L be drawn at Right Angles to A G, and A L be joined: Then, biſecting the Circumference E B, as alſo the Half thereof, and doing thus continually, we ſhall have left, at laſt, a certain Circumference leſs than that Part of the Circumference of the Circle B L, which is ſubtended by a Right Line equal to G L. Let this be the Circumference B K; then the Right Line B K is leſs than * 16 of this. G L; and B K ſhall be the * Side of a Polygon of equal Sides, even in Number, not touching the leſſer Circle: Now, let the Sides of the Polygon, in the Quadrant of the Circle B E, be the Right Lines B K, K L, D M, M E; and produce the Line joining the ‡ 12. 11. Points K, A, to N; and raiſe ‡ A X from the Point A, perpendicular to the Plane of the Circle B C D E, meeting the Superficies of the Sphere in the Point X; and let Planes be drawn thro' A X and B D, and thro'

A X and K N; which, from what has been ſaid, will
make great Circles in the Superficies of the Sphere;
and let B X D, K X N, be Semicircles on the Diameters
B D, K N: Then, becauſe X A is perpendicular
to the Plane of the Circle B C D E, all Planes
that paſs thro' X A ſhall alſo * be perpendicular to * 18. 11.
that ſame Plane. Therefore the Semicircles B X D,
K X N, are perpendicular to that ſame Plane. And
becauſe the Semicircles B E D, B X D, K X N, are
equal; for they ſtand upon equal Diameters B D,
K N; their Quadrants B E, B X, K X, ſhall be alſo
equal. And therefore, as many Sides as the Polygon
in the Quadrant B E has, ſo many Sides may there
be in the Quadrants B X, K X, equal to the Sides
B K, K L, L M, M E. Let thoſe Sides be B O,
O P, P R, R X, K S, S T, T Y, Y X; and join
S O, T P, Y R; and let Perpendiculars be drawn
from O, S, to the Plane of the Circle B C D E:
Theſe will † fall on B D, K N, the common Sections † 38. 11.
of the Planes; becauſe the Planes of the Semicircles
B X D, K X N, are perpendicular to the Plane of the
Circle B C D E: Let the ſaid Perpendicular be O V,
S Q, and join V Q; then, ſince the equal Circumferences
B O, S K, are taken in the equal Semicircles
B X D, K X N, and O V, S Q, are Perpendiculars;
O V ſhall be equal to S Q, and B V to K Q. But
the Whole B A is equal to the Whole K A; therefore
the Part remaining V A is equal to the Part remaining
Q A: Therefore, as B V is to V A, ſo is
K Q to Q A; and ſo V Q is ‡ parallel to B K. And ‡ 2. 6.
ſince O V and S Q are both perpendicular to the Plane
of the Circle B C D E, O V ſhall be * parallel to S Q. * 6. 11.
But it has alſo been proved equal to it; wherefore
Q V, S O, are † equal and parallel. And becauſe † 33. 1.
Q V is parallel to S O, and alſo parallel to K B; S O
ſhall be alſo ‡ parallel to K B: Join B O, K S, and ‡ 9. 11.
then, K B O S is * a quadrilateral Figure in one Plane: * 7. 11.
For if two Right Lines be parallel, and Points be
taken in both of them, a Right Line joining the ſaid
Points is in the ſame Plane as the Parallels are. And
for the ſame Reaſon, each of the quadrilateral Figures
S O P T, T P R Y, are in one Plane. And the
Triangle Y R X is † in one Plane; therefore, if Right † 2. 11.
Lines be ſuppoſed to be drawn from the Points O, S,

P, T, R, Y, to the Point A, there will be conſtituted a certain ſolid polyhedrous Figure within the Circumferences B X, K X, compoſed of Pyramids, whoſe Baſes are the quadrilateral Figures K B O S, S O P T, T P R Y, and the Triangle Y R X; and Vertices the Point A. And if there be made the ſame Conſtruction on each of the Sides K L, L M, M E, like as we have done on the Side K B; and alſo in the other three Quadrants, and the other Hemiſphere, there will be conſtituted a polyhedrous Figure deſcribed in the Sphere, compoſed of Pyramids whoſe Baſes will be equal and ſimilar to the aforeſaid quadrilateral Figures, and Triangle Y R X, and Vertices the Point A. Now, I ſay, the ſaid Polyhedron does not touch the Superficies of the Sphere, wherein
‡ 11. 11. the Circle F G H is. Let A Z be drawn ‡ from the Point A, perpendicular to the Plane of the quadrilateral Figure K B S O, meeting it in the Point Z; and join B Z, Z K: Then, ſince A Z is perpendicular to the Plane of the quadrilateral Figure K B S O, it ſhall alſo
*Def. 3. 11. be * perpendicular to all Right Lines that touch it, and are in the ſame Plane: Wherefore A Z is perpendicular to B Z and Z K. And becauſe A B is equal to A K, the Square of A B ſhall be alſo equal to the
† 4. 17. Square of A K; and the Squares of A Z, Z B, are † equal to the Square of A B; for the Angle at Z is a Right Angle; and the Squares of A Z, Z K, are equal to the Square of A K: Therefore the Square of A Z, Z B, are equal to the Squares of A Z, Z K. Let the common Square of A Z be taken away, and then the Square of B Z, remaining, is equal to the Square of Z K, remaining; and ſo the Right Line B Z is equal to the Right Line Z K. After the ſame manner we demonſtrate, that Right Lines drawn from the Point Z to the Point O, S, are each equal to B Z, Z K. Therefore a Circle deſcribed about the Centre Z, with either of the Diſtances, Z B, Z K, will alſo paſs thro' the Points O, S. And, becauſe B K S O is a quadrilateral Figure in a Circle, and O B, B K, K S, are equal; and O S is leſs than B K; the Angles B Z K ſhall be obtuſe; and ſo B K greater than B Z. But G L alſo is greater than B K; therefore G L is much greater than B Z; and the Square of G L is greater than the Square of B Z. And ſince

A L is equal to A B, the Square of A L ſhall be equal to the Square of A B. But the Squares of A G, G L, together, are equal to the Square of A L; and the Squares of B Z, Z A, together, equal to the Square of A B: Therefore the Squares of A G, G L, together, are equal to the Squares of B Z, Z A, together. But the Square of B Z is leſs than the Square of G L; therefore the Square of Z A is greater than the Square of A G; and ſo the Right Line Z A will be greater than the Right Line A G. But A Z is perpendicular to one Baſe of the Polyhedron, and A G reaches to the Superficies of the leſſer Sphere: Wherefore the Polyhedron does not touch the Superficies of the leſſer Sphere. Therefore, *there is deſcribed a ſolid Polyhedron in the greater of two Spheres, having the ſame Centre, which doth not touch the Superficies of the leſſer Sphere;* which was to be done.

Coroll. Alſo, if a ſolid Polyhedron be deſcribed in ſome other Sphere, ſimilar to that which is deſcribed in the Sphere B C D E; the ſolid Polyhedron deſcribed in the Sphere B C D E, to the ſolid Polyhedron deſcribed in that other Sphere, ſhall have a triplicate Proportion of that which the Diameter of the Sphere B C D E hath to the Diameters of that other Sphere. For, the Solids being divided into Pyramids equal in Number, and of the ſame Order, the ſame Pyramids ſhall be ſimilar. But ſimilar Pyramids are to each other in a triplicate Proportion of their homologous Sides; therefore the Pyramid, whoſe Baſe is the quadrilateral Figure K B O S, and Vertex the Point A, to the Pyramid of the ſame Order in the other Sphere, has a triplicate Proportion of that which the homologous Side of the one has to the homologous Side of the other; that is, which A B, drawn from the Centre A of the Sphere, to that Line which is drawn from the Centre of the other Sphere. In like manner, every one of the Pyramids, that are in the Sphere whoſe Centre is A, to every one of the Pyramids of the ſame Order in the other Sphere, hath a triplicate Proportion of that which A B has to that Line drawn from the Centre of the other Sphere: And as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents. Wherefore the whole ſolid Polyhedron, which is in the Sphere deſcribed about the Centre A, to the whole ſolid Polyhedron that is in the other Sphere, hath a triplicate Proportion of that which A B hath to the Line drawn from the Centre of the other Sphere; that is, which the Diameter B D has to the Diameter of the other Sphere.

PROPOSITION XVIII.

Theorem.

Spheres are to one another in a triplicate Proportion of their Diameters.

SUppoſe A B C, D E F, are two Spheres, whoſe Diameters are B C, E F. I ſay, the Sphere A B C to the Sphere D E F, has a triplicate Proportion of that which B C has to E F.

For, if be not ſo, the Sphere A B C to a Sphere either leſſer or greater than D E F, will have a triplicate Proportion of that which B C has to E F. Firſt, let it be to a leſſer, as G H K; and ſuppoſe the Sphere D E F to be deſcribed about the Sphere G H K; and let there be deſcribed * a ſolid Polyhedron in the greater Sphere D E F, not touching the Superficies of the leſſer Sphere G H K; alſo, let a ſolid Polyhedron be deſcribed in the Sphere A B C, ſimilar to that which is deſcribed in the Sphere D E F; then the ſolid Polyhedron in the Sphere A B C, to the ſolid Polyhedron in the Sphere D E F, will have a † triplicate Proportion of that which B C has to E F: But the Sphere A B C to the Sphere G H K, hath a triplicate Proportion of that which B C hath to E F; therefore, as the Sphere A B C is to the Sphere G H K, ſo is the ſolid Polyhedron in the Sphere A B C, to the ſolid Polyhedron in the Sphere D E F; and (by Inverſion) as the Sphere A B C is to the ſolid Polyhedron that is in it, ſo is the Sphere G H K to the ſolid Polyhedron that is in the Sphere D E F. But the Sphere A B C is greater than the Solid Polyhedron that is in it; therefore the Sphere G H K is alſo greater than the Solid Polyhedron that is in the Sphere D E F, and

* *17 of this.*

† *Cor. to the laſt Prop.*

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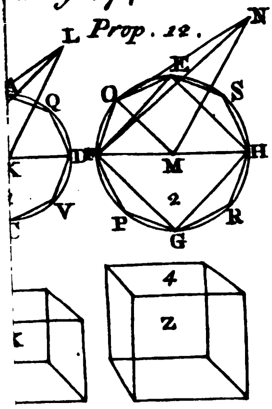

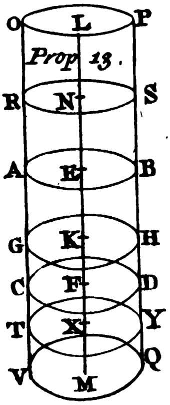

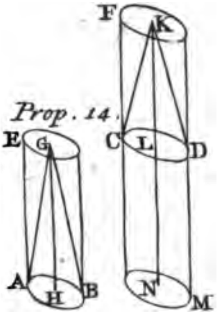

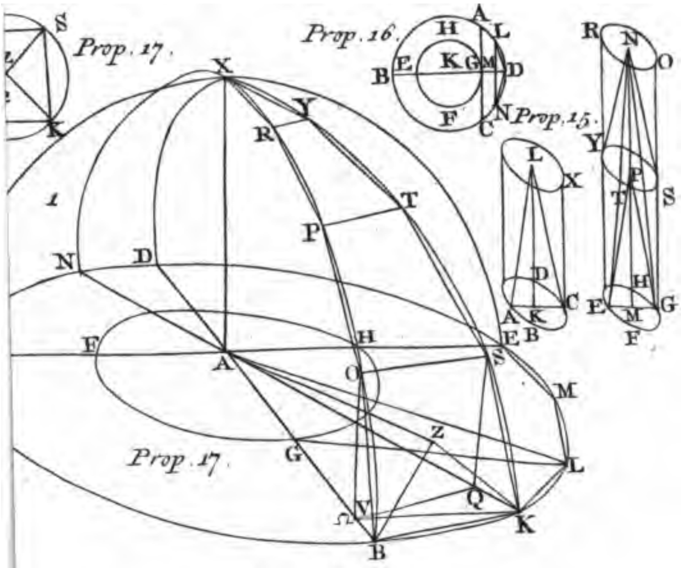

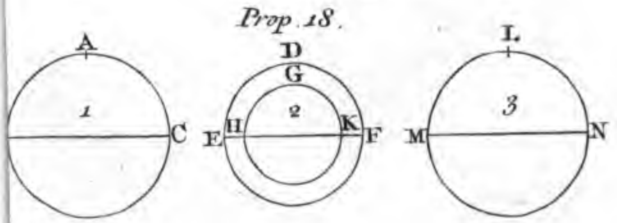

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and alſo leſs than it, as being comprehended thereby; which is abſurd; therefore the Sphere A B C to a Sphere leſs than the Sphere D E F, hath not a triplicate Proportion of that which B C has to E F. After the ſame manner it is demonſtrated, that the Sphere D E F to a Sphere leſs than A B C, has not a triplicate Proportion of that which E F has to B C: I ſay, moreover, that the Sphere A B C to a Sphere greater than D E F, hath not a triplicate Proportion of that which B C has to E F: For, if it be poſſible, let it have to the Sphere L M N greater than D E F; then (by Inverſion) the Sphere L M N to the Sphere A B C, ſhall have a triplicate Proportion of that which the Diameter E F has to the Diameter B C. But as the Sphere L M N is to the Sphere A B C, ſo is the Sphere D E F to ſome Sphere leſs than A B C, becauſe the Sphere L M N is greater than D E F. Therefore the Sphere D E F to a Sphere leſs than A B C, hath a triplicate Proportion of that which E F has to B C, which is abſurd, as has been before proved. Therefore the Sphere A B C to a Sphere greater than D E F, has not a triplicate Proportion of that which B C has to E F. But it has alſo been demonſtrated, that the Sphere A B C to a Sphere leſs than D E F, has not a triplicate Proportion of that which B C has to E F: Therefore, *the Sphere* A B C *to the Sphere* D E F, *has a triplicate Proportion of that which* B C *has to* E F; which was to be demonſtrated.

FINIS.

THE ELEMENTS

Of PLANE and SPHERICAL.

TRIGONOMETRY.

DEFINITIONS.

THE Buſineſs of Trigonometry is, to find the Angles when the Sides are given, and the Sides, or the Ratios of the Sides, when the Angles are given; and to find Sides and Angles, when Sides and Angles are given: In order to which, it is neceſſary, that not only the Peripheries of Circles, but alſo certain Right Lines in and about Circles, be ſuppoſed divided into ſome determined Number of Parts.

And ſo the antient Mathematicians thought fit to divide the Periphery of a Circle into 360 *Parts, which they call Degrees; and every Degree into* 60 *Minutes; and every Minute into* 60 *Seconds; and, again, every Second into* 60 *Thirds; and ſo on. And every Angle is ſaid to be of ſuch a Number of Degrees and Minutes, as there are in the Arc meaſuring that Angle.*

There are ſome that would have a Degree divided into centeſimal Parts, rather than ſexageſimal ones; and perhaps it would be more uſeful to divide, not only a Degree, but even the whole Circle, into a duplicate Ratio; which Diviſion may ſome Time or other gain Place. Now, if a Circle contains 360 *Degrees, a Quadrant thereof, which is the Meaſure of a Right Angle, will* *be* 90 *of thoſe Parts: And if it contains* 100 *Parts, a Quadrant will be* 25 *of theſe Parts.*

The Complements *of an Arc is the Difference thereof from a Quadrant.*

A Chord *or* Subtenſe, *is a Right Line drawn from one End of the Arc to the other.*

The Right Sine *of any Arc, which is alſo commonly called only a* Sine, *is a Right Line drawn, from one End of an Arc, perpendicular to the Radius drawn thro' the other End of the ſaid Arc; and is therefore the Semi-ſubtenſe of double the Arc*; viz. $DE = \frac{1}{2} DO$, *and the Arc* DO *is double of the Arc* DB. *Hence, the Sine of an Arc of* 30 *Degree is equal to one half of the Radius. For* (*by* Corol. 15. El. 4.) *the Side of an Hexagon inſcribed in a Circle, that is, the Subtenſe of* 60 *Degrees, is equal to the Radius. A Sine divides the Radius into two Segments* CE, EB; *one of which,* CE, *which is intercepted between the Centre and the Right Sine, is the Sine of the Complement of the Arc* DB *to a Quadrant* (*for* $CE = FD$, *which is the Sine of the Arc* DH), *and is called the* Coſine: *The other Segment* BE, *which is intercepted between the Right Sine and the Periphery, is called a* Verſed Sine, *and ſometimes a* Sagitta.

And, if the Right Line CG *be produced from the Centre* C, *thro' one End* D *of the Arc, until it meets the Right Line* BG, *which is perpendicular to the Diameter drawn thro' the other End* B *of the Arc; then* CG *is called the* Secant, *and* BG *the* Tangent, *of the Arc* DB.

The Coſecant *and* Cotangent *of an Arc are the Secant and Tangent of that Arc which is the Complement of the former Arc to a Quadrant.* Note, *As the Chord of an Arc, and of its Complement to a Circle, is the ſame; ſo, likewiſe, are the Sine, Tangent, and Secant, of an Arc, the ſame as the Sine, Tangent, and Secant, of its Complement to a Semicircle.*

The Sinus Totus *is the greateſt Sine, or the Sine of* 90 *Degrees, which is equal to the Radius of the Circle.*

A Trigonometrical Canon *is a Table, which, beginning from one Minute, orderly expreſſes the Lengths that every Sine, Tangent, and Secant have, in reſpect of the Radius, which is ſuppoſed Unity; and is conceived to be divided into* 10,000,000 *or more decimal Parts. And ſo the Sine, Tangent, or Secant, of an Arc, may be had* *by Help of this Table; and, contrariwiſe, a Sine, Tangent, or Secant, being given, we may find the Arc it expreſſes.* Take notice, *That in the following Tract,* R. *ſignifies the Radius,* S. *a Sine,* Coſ. *a Coſine,* T. *a Tangent, and* Cot. *a Cotangent; alſo* A C q *ſignifies the Square of the Right Line* A C; *and the Marks or Characters* +, —, =, : : :, *and* √, *are, ſeverally, uſed to ſignify Addition, Subtraction, Equality, Proportionality, and the Extraction of the Square Root: Again, when a Line is drawn over the Sum or Difference of two Quantities, then that Sum or Difference is to be conſidered as one Quantity.*

The CONSTRUCTIONS of the *Trigonometrical Canon.*

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is alſo given.

FOR (by 47 of the firſt *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$ and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the Square Root, there is given $AC = \surd\overline{ABq + BCq}$; and $AB = \surd\overline{ACq - BCq}$; and $BC = \surd\, ACq - ABq$.

PROPOSITION II.

PROBLEM.

The Sine D E *of the Arc* B D, *and the Radius* C D, *being given, to find the Coſine* D F.

THE Radius C D, and the Sine D E, being given in the Right-angled Triangle C D E, there will be given (by the laſt *Prop.*) $\surd\overline{CDq - DEq} = (CE =)\ DF$.

PROPOSITION III.

PROBLEM.

The Sine D E *of any Arc* D B *being given, to find* D M *or* B M, *the Sine of Half the Arc.*

D E being given, C E (by the laſt *Prop.*) will be given, and accordingly E B, which is the Difference between the Coſine and Radius. Therefore D E, E B, being given, in the Right-angled Triangle D B E, there will be given D B, whoſe Half D M is the Sine of the Arc D L $=\frac{1}{2}$ the Arc B D.

PROPOSITION IV.

PROBLEM.

The Sine B M *of the Arc* B L *being given, to find the Sine of double that Arc.*

THE Sine B M being given, there will be given (by *Prop.* 2.) the Coſine C M. But the Triangles C B M, D B E, are equiangular, becauſe the Angles at E and M are Right Angles, and the Angle at B common: Wherefore (by 4. 6.) we have C B : C M : : (B D, or) 2 B M : D E. Whence, ſince the three firſt Terms of this Analogy are given, the fourth alſo, which is the Sine of the Arc D B, will be known.

Coroll. Hence C B : 2 C M : : B D : 2 D E; that is, the Radius is to double the Coſine of one Half of the Arc D B, as the Subtenſe of the Arc D B is to the Subtenſe of double that Arc. Alſo, C B : 2 C M : : (2 B M : 2 D E : :) B M : D E : : $\frac{1}{2}$ C B : C M. Wherefore the Sine of an Arc, and the Sine of its Double, being given, the Coſine of the Arc itſelf is given.

PRO-

PROPOSITION V.

PROBLEM.

The Sines of two Arcs, B D, F D, *being given, to find* F I *the Sine of the Sum, as likewiſe* E L, *the Sine of their Difference.*

LET the Radius C D be drawn, and then C O is the Coſine of the Arc F D, which accordingly is given, and draw O P thro' O parallel to D K; alſo let O M, G E, be drawn parallel to C B: Then, becauſe the Triangles C D K, C O P, C H I, F O H, F O M, are equiangular; in the firſt Place C D : D K : : C O : O P, which, conſequently, is known. Alſo, we have C D : C K : : F O : F M; and ſo, likewiſe, this ſhall be known. But becauſe F O = E O, then will F M = M G = O N; and ſo O P + F M = F I = Sine of the Sum of the Arcs: And O P — F M; that is, O P — O N = E L = Sine of the Difference of the Arcs; which were to be found.

Coroll. Becauſe the Differences of the Arcs B E, B D, B F, are equal, the Arc B D ſhall be an Arithmetical Mean between the Arcs B E, B F.

PROPOSITION VI.

THEOREM.

The ſame Things being ſuppoſed, the Radius is to double the Coſine of the mean Arc, as the Sine of the Difference is to the Difference of the Sines of the Extremes.

FOR we have C D : C K : : F O : F M; whence, by doubling the Conſequents, C D : 2 C K : : F O : (2 F M, or) to F G, which is the Difference of the Sines E L, F I. W. W. D.

Coroll. If the Arc B D be 60 Degrees, the Difference of the Sines F I, E L, ſhall be equal to the Sine F O, of the Difference. For, in this Caſe, C K is the Sine of 30 Degrees; the double whereof is equal to the Radius; and ſo, ſince C D = 2 C K, we ſhall have

FO=FG. And, conſequently, if the two Arcs BE, BF, are equidiſtant from the Arc of 60 Degrees, the Difference of the Sines ſhall be equal to the Sine of the Difference FD.

Coroll. 2. Hence, if the Sines of all Arcs diſtant from one another by a given Interval, be given, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees=the Sine of 59 Degrees+the Sine of 1 Degree; and the Sine of 62 Degrees=the Sine of 58 Degrees+the Sine of 2 Degrees: Alſo, the Sine of 63 Degrees=the Sine of 57 Degrees+the Sine of 3 Degrees, and ſo on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, diſtant from each other by a given Interval, be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the precedent Analogy, all the Sines to 30 Degrees may be found: For the Radius is to double the Coſine of 15 Degrees, as the Sine of 1 Degree is to the Difference of the Sines of 14 Degrees, and 16 Degrees: So, alſo, is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and ſo on continually, until you come to the Sine of 30 Degrees.

After the ſame manner, as the Radius is to double the Coſine of 30 Degrees, or to double the Sine of 60 Degrees, ſo is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees :: Sine 2 Degrees to the Difference of the Sines of 28 and 32 Degrees :: Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But, in this Caſe, the Radius is to double the Coſine of 30 Degrees, as 1 to $\sqrt{3}$.

See FIG. *of the* DEFINITIONS.

* Let BD be an Arc of 30 Degrees:

Rad. Tan. Coſine Sine

Then, as $\underline{CB} : \underline{BG} :: \underline{FD} : \underline{DE}$. $\underline{DO} = CB$; *ergo* $DE = \frac{1}{2}$; $\sqrt{CDq - DEq} = CE = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}$; $CD : CE :: 1 : \sqrt{\frac{3}{4}}$; $CD : 2CE :: 1 : 2\sqrt{\frac{3}{4}} = \sqrt{\frac{12}{4}}\sqrt{3}$. *Q. E. D.*

And, accordingly, if the Sines of the Diſtances from the Arc of 30 Degrees, be multiplied by $\sqrt{3}$, the Differences of the Sines will be had.

So, likewiſe, may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Coſines of one and two Minutes given. For, as the Radius is to double the Coſine of 2′ : : Sine 1′ : Difference of the Sines of 1′ and 3′ : : Sine 2′ : Difference of the Sines of 0′ and 4′; that is, to the Sine of 4′. And ſo, the Sines of the four firſt Minutes being given, we may thereby find the Sines of the others to 8′, and from thence to 16′, and ſo on.

PROPOSITION VII.

THEOREM.

In ſmall Arcs, the Sines and Tangents of the ſame Arcs, are nearly to one another, in a Ratio of Equality.

FOR, becauſe the Triangles C E D, C B G, are equiangular, C E : C B : : E D : B G. But as the Point E approaches B, E B will vaniſh in reſpect of the Arc B D; whence C E will become nearly equal to C B, and ſo E D will be alſo nearly equal to B G. If E B be leſs than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be alſo leſs than the $\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is leſs than the Tangent, and greater than its Sine, and the Sine and Tangent of a very ſmall Arc are nearly equal; it follows, that the Arc ſhall be nearly equal to its Sine: And ſo, in very ſmall Arcs, it ſhall be, as Arc is to Arc, ſo is Sine to Sine

PROPOSITION VIII.

PROBLEM.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inſcribed in a Circle, that is, the Subtenſe of 60 Degrees, is equal to the Radius (*by Coroll.* 15*th of the* 4*th*); and ſo the Half of the Radius ſhall be the Sine of the Arc of 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found (*by Prop.* 3.) Alſo the Sine of the Arc of 15 Degrees being given (*by the ſame Prop.*) we may have the Sine of 7 Degrees 30 Minutes: So, likewiſe, can we find the Sine of the Half of this *viz.* 3 Degrees $45'$; and ſo on, until twelve Biſections being made, we come to an Arc of 52^{2}, 44^{3}, 03^{4}, 45^{5}, whoſe Coſine is nearly equal to the Radius; in which Caſe (*as is manifeſt from Prop.* 7.) Arcs are proportional to their Sines: And ſo, as the Arc of 52^{2}, 44^{3}, 03^{4}, 45^{5}, is to an Arc of one Minute, ſo ſhall the Sine before found be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop.* 2. *and* 4.) the Sine and Coſine of two Minutes will be had.

PROPOSITION IX.

THEOREM.

If the Angle B A C, *being in the Periphery of a Circle, be biſected by the Right Line* A D, *and if* A C *be produced until* D E=A D *meets it in* E; *then ſhall* C E=A B.

IN the quadrilateral Figure A B D C (*by* 22. 3.) the Angles B and D C A are equal to two Right Angles =D C E+D C A (*by* 13. 1.): Whence the Angle B =D C E. But, likewiſe, the Angle E=D A C (*by* 5. 1.)=D A B, and D C=D B: Wherefore the Triangles B A D and C E D are congruous, and ſo C E is equal to A B. W. W. D.

PROPOSITION X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, *&c. be equal; and let the Subtenſes of the Arcs* AB, AC, AD, AE, *&c. be drawn; then will* AB : AC :: AC : AB+AD :: AD : AC+AE :: AE : AD+AF :: AF : AE+AG.

LET AD be produced to H, AE to I, AF to K, and AG to L, ſo that the Triangles ACH, ADI, AEK, AFL, be Iſoſceles ones: Then, becauſe the Angle BAD is biſected, we ſhall have DH=AB (*by the laſt Prop.*); ſo likewiſe ſhall EI=AC, FK=AD, alſo GL=AE.

But the Iſoſceles Triangles ABC, ACH, ADI, AEK, AFL, becauſe of the equal Angles at the Baſes are equiangular: Wherefore it ſhall be, as AB : AC :: AC : (AH=) AB+AD :: AD : (AI=) AC+AE :: AE : (AK=) AD+AF :: AF : (AL=) AE+AG. W. W. D.

Coroll. 1. Becauſe AB is to AC, as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, it ſhall alſo be (*by Coroll. Prop.* 4.) as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, ſo is $\frac{1}{2}$AB : $\frac{1}{2}$AC :: $\frac{1}{2}$AC : $\frac{1}{2}$AB+$\frac{1}{2}$AD :: $\frac{1}{2}$AD : $\frac{1}{2}$AC+$\frac{1}{2}$AE :: $\frac{1}{2}$AE : $\frac{1}{2}$AD+$\frac{1}{2}$AF, *&c.* Now let each of the Arcs AB, BC, CD, *&c.* be $2'$; then will $\frac{1}{2}$AB be the Sine of one Minute, $\frac{1}{2}$AC the Sine of 2 Minutes, $\frac{1}{2}$AD the Sine of 3 Minutes, $\frac{1}{2}$AE the Sine 4 Minutes, *&c.* Whence, if the Sines of one and two Minutes be given, we may eaſily find all the other Sines, in the following manner.

Let the Coſine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. $59'$, be called Q; and make the following Analogies; R. : 2 Q :: Sin. $2'$: S $1'$+S. $3'$. Wherefore the Sine of 3 Minutes will be given. Alſo, R. : 2 Q :: S. $3'$: S. $2'$+S. $4'$. Wherefore the S. $4'$ is given. And R. : 2 Q :: S. $4'$: S. $3'$+S. $5'$; and ſo the Sine of $3'$ will be had.

Likewiſe, R. : 2 Q : : S. 5′ : S. 4′+S. 6′; and ſo we ſhall have the Sine of 6′. And, in like manner, the Sines of every Minute of the Quadrant will be given. And becauſe the Radius, or the firſt Term of the Analogy, is Unity, the Operations will be with great Eaſe and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only (*by Cor.* 1. *Prop.* 6.)

The Sines being given, the Tangents and Secants may be found from the following Analogies (in the Figure for the Definitions); becauſe the Triangles CED, CBG, CHI, are equiangular, we have

CE : ED : : CB : BG; that is, Coſ. : S. : : R. : T.
GB : BC : : CH : HI; that is, T. : R : : R. : Cot.
CE : CD : : CB : CG; that is, Coſ. : R. : : R. : Secant,
DE : CD : : CH : CI; that is, S. : R. : : R. : Coſec.

SCHOLIUM.

That great Geometrician, and incomparable Philoſopher, Sir Iſaac Newton, *was the firſt that laid down a Series converging* in infinitum; *from which, having the Arcs given, their Sines may be found. Thus, if an Arc be called* A, *and the Radius be an Unit, the Sine thereof will be found to be*

$$A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9} \;\&c.$$

And the Coſine,

$$1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8} \;\&c.$$

Theſe Series in the Beginning of the Quadrant, when the Arc A *is but ſmall, ſoon converge. For in the Series for the Sine, if* A *does not exceed* 10 *Minutes, the two firſt Terms thereof,* viz. $A - \frac{1}{6}A^3$, *give the Sine to* 15 *Places of Figures. If the Arc* A *be not greater than one Degree, the three firſt Terms will exhibit the Sine to* 15 *Places of Figures; and ſo the ſaid Series are very uſeful for finding the firſt and laſt Sines of the Quadrant. But the greater the Arc* A *is, the more are the Terms of the Series required to have the Sine, in Num-*

Numbers, true to a given Place of Figures. And then, when the Arc is nearly equal to the Radius, the Series converges very ſlow, and therefore, to remedy this, I have deviſed other Series, ſimilar to the Newtonian *ones, wherein I ſuppoſe, the Arc, whoſe Sine is ſought, is the Sum or Difference of two Arcs, viz.* A+z, *or* A—z: *And let the Sine of the Arc* A *be called* a, *and the Coſine* b. *Then the Sine of the Arc* A+z *will be expreſſed thus:*

$$1.\quad a+\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}\ \&c.$$

And the Coſine is

$$2.\quad b-\frac{az}{1}-\frac{bz^2}{1.2}+\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}-\frac{az^5}{1.2.3.4.5}-\frac{bz^6}{1.2.3.4.5.6}$$

In like manner the Sine of the Arc A—z *is*

$$3.\quad a-\frac{bz}{1}-\frac{az^2}{1.2}+\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}-\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}$$

And the Coſine is

$$4.\quad b+\frac{az}{1}-\frac{bz^2}{1.2}-\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}+\frac{az^5}{1.2.3.4.5}\ \&c.$$

The Arc A, *is an arithmetical Mean between the Arc* A—z, *and the Arc* A+z. *And the Differences of the Sines are*

$$5.\quad \frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

$$6.\quad \frac{bz}{1}+\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}-\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}+\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

Whence the Difference of the Differences, or ſecond Difference, will be

$$7.\quad \frac{2az^2}{1.2}-\frac{2az^4}{1.2.3.4}+\frac{2az^6}{1.2.3.4.5.6}\ \&c.$$

$$\text{Or } 2a\times\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}\ \&c.$$

Which Series is equal to double the Sine of the mean Arc, drawn into the Verſed Sine of the Arc z, *and converges very ſoon. So that if* z *be the firſt Minute of the Quadrant, the firſt Term of the Series gives the ſecond Difference to* 15 *Places of Figures, and the ſecond Term to* 25 *Places.*

From hence, if the Sines of the Arcs, diſtant one Minute from each other, be given; the Sines of all the Arcs, that are in the ſame Progreſſion, may be found by an exceeding eaſy Operation.

In the firſt and ſecond Series, if A=0; *then ſhall* a=0, *and* b *its Coſine will become Radius, or* 1. *And hence if the Terms wherein* a *is, are taken away, and* 1 *be put inſtead of* b, *the Series will become the* Newtonian. *In the third and fourth Series, if* A *be* 90 *Degrees, we ſhall have* b=0, *and* a=1. *Whence, again, taking away all the Terms wherein* b *is, and putting* 1 *inſtead of* a, *we ſhall have the* Newtonian *Series ariſe.*

Note, *All the ſaid Series eaſily flow from the* Newtonian *ones. By the fifth* Propoſition.

PROPOSITION XI.

THEOREM.

In a Right-angled Triangle, if the Hypothenuſe be made the Radius, then are the Sides the Sines of their oppoſite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of its oppoſite Angle, and the Hypothenuſe is the Secant of that Angle.

IT is manifeſt, that C B is the Sine of the Arc D, and A B the Coſine thereof; but the Arc C D is the Meaſure of the Angle A, and the Complement of the Meaſure of the Angle C: Moreover, if A B in the ſecond Figure to this Propoſition, be ſuppoſed Radius, then B C is the Tangent, and A C the Secant of the Arc B D, which is the Meaſure of the Angle A. So, alſo, if B C be made the Radius, then is B A the Tangent, and A C the Secant, of the Arc B E, or Angle C. W.W.D. Therefore, as A C being taken as ſome given Meaſure, is to B C taken in the ſame Meaſure; ſo ſhall the Number 10,000,000 Parts, into which the Radius is ſuppoſed to be divided, be to a Number expreſſing, in the ſame Parts, the Length of the Sine of the Angle A; that is,

it will be, as AC : BC : : R : S, A.
by the ſame Reaſon, as AC : BA : : R : S, C.
alſo, as AB : BC : : R : T, A.
and, as BC : BA : : R : T, C.

And ſo, if any three of theſe Proportionals be given, the fourth may be found by the *Rule of Three*.

PROPOSITION XII.

THEOREM.

Sides of plane Triangles are as the Sines of their oppoſite Angles.

IF the Sides of a Triangle, inſcribed in a Circle, be biſected by perpendicular Radii; then ſhall the half Sides be the Sines of the Angles at the Periphery; for the Angle B D C, at the Centre, is double of the Angle B A C at the Periphery (*by* 20 *El. lib.* 3.); and ſo (the Half of B D C, *viz.*) B D E = B A C, and B E is the Sine of (B D E, or) B A C. For the ſame Reaſon, B F ſhall be the Sine of the Angle B C A, and A G the Sine of the Angle A B C.

In a Right-angled Triangle we have $BD = \frac{1}{2}BC$ = Radius (*by* 31 *Eucl.* 3.); but Radius is the Sine of a Right Angle: Whence half B C is the Sine of the Angle A.

In an obtuſe-angled Triangle, let BI, CI, be drawn; and then the Angle I ſhall be the Complement of the Angle A to two Right Angles (*by* 22 *El.* 3.); and ſo they ſhall both have the ſame Sine. But the Angle B D E (whoſe Sine is B E) = Angle I; therefore B E ſhall be the Sine of the Angle B A C. And ſo in every Triangle, the Halves of the Sides are the Sines of the oppoſite Angles: But it is manifeſt, that the Sides are to one another as their Halves. And therefore, *the Sides of plane Triangles are as the Sines of their oppoſite Angles.* W. W. D.

PROPOSITION XIII.

THEOREM.

In a plane Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Baſe, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle A B C, whoſe Legs are A B, B C, and Baſe A C. Produce A B to H, ſo that B H=B C; then ſhall A H be the Sum of the Legs; and if you make B I=B A, then I H will be the Difference of the Legs. Alſo, the Angle H B C=Angles A+A C B (*by* 32 *El.* 1.); and ſo E B C the Half thereof =half the Sum of the Angles A, and A C B, and its Tangent (putting the Radius=E B) is E C. Again, let B D be drawn parallel to A C, and make H F=C D; then, ſince H B=C B, we ſhall have (*by* 4 *El.* 1.) the Angle H B F=C B D=B C A (*by* 29 *El.* 1.) Alſo, the Angle H B D=Angle A; whence F B D ſhall be the Difference of the Angles A and A C B, and E B D, whoſe Tangent is E D, half their Difference. Let I G be drawn thro' I parallel to A C or B D; and then (*by* 2 *El.* 6.) A B : B I : : C D : D G. But A B=B I; whence we ſhall have C D=D G; but C D=H F, and ſo H F=D G; and, conſequently, H G=D F, and $\frac{1}{2}$ H G=$\frac{1}{2}$ D F=D E; and becauſe the Triangles A H C, I H G, are equiangular, it ſhall be, as A H : I H : : H C : H G : : $\frac{1}{2}$ H C : $\frac{1}{2}$ H G : : E C : E D. That is, A H *the Sum of the Legs, to* I H *the Difference of the Legs, ſhall be, as* E C *the Tangent of one half the Sum of the Angles at the Baſe, to* E D *the Tangent of one half of their Difference.* W. W. D.

PROPOSITION XIV.

THEOREM.

In a plane Triangle, the Baſe, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Baſe, are proportional.

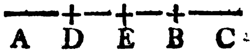
LET D C be the Baſe of the Triangle B C D. About the Centre B, with the Radius B C, let a Circle be deſcribed; produce D B to G, and from B let fall B E perpendicular to the Baſe; then ſhall D G=D B+B C=Sum of the Sides, and D H=Difference of the Sides; and D E, C E, are the Segments of the Baſe whoſe Difference is D F; then, becauſe (*by Cor. Prop.* 37. *El.* 3.) the Rectangle under D C and D F is equal to the Rectangle under D G, D H, it ſhall be (*by* 16 *El.* 6.), as D C : D G : : D H : D F.

PROPOSITION XV.

THEOREM.

The Sum and Difference of any two Quantities being given, to find the Quantities themſelves.

IF one half of the Sum be added to one half of the Difference, the Aggregate ſhall be equal to the greater of the Quantities; and if from one half of the Sum be taken one half of the Difference, the Reſidue ſhall be equal to the leſſer of the Quantities. For, let there be two Quantities A B, B C; and let there be taken A D=B C; then D B will be their Difference, and A C their Sum; which, biſected in E, gives A E or E C the half Sum; and D E or E B the half Difference. Hence, A B=A E+E B=the half Sum+the half Difference; and B C=C E—E B= the half Sum—the half Difference.



Note, I. *In any plane Triangle, if two Angles be given the third Angle is alſo given; becauſe it is their Complement to two Right Angles.*

II. *If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, becauſe it is the Complement of the given Angle to a Right Angle.*

III. *And if two Sides of a Right-angled Triangle be given, the other Side may be found by the firſt Propoſition, without a Canon.*

The Trigonometrical Solutions of a Right-angled Triangle may be as follows, Vid. Fig. A.

	Given	Sought	Make, as
1	The Legs AB and BC	The Angles.	AB : BC : : R : T, of the Angle A, whoſe Complement is the Angle C.
2	The Leg AB, and the Hypothenuſe AC	The Angles.	AC : AB : : R : S, C, whoſe Complement is the Angle A.
3	The Leg AB, and the Angle A.	The other Leg BC, and the Hypothenuſe AC.	R : T, A : : AB : BC. S, C : R : : AB : AC.
4	The Hypothenuſe AC, and the Angle C.	The Leg AB.	R : S, C : : AC : AB.

The Trigonometrical Solutions of Oblique-angled Triangles. Vid. Fig. to Prop. 12.

	Given	Sought	Make, as
1	The Angles A, B, C, and the Side AB	The Sides BC and AC.	S, C : S, A : : AB : BC. Alſo, S, C : S, B : : AB : AC. But when two Angles are given, the third is alſo given: Whence the Caſe wherein two Angles and a Side are given, to find the reſt, falls into this Caſe.

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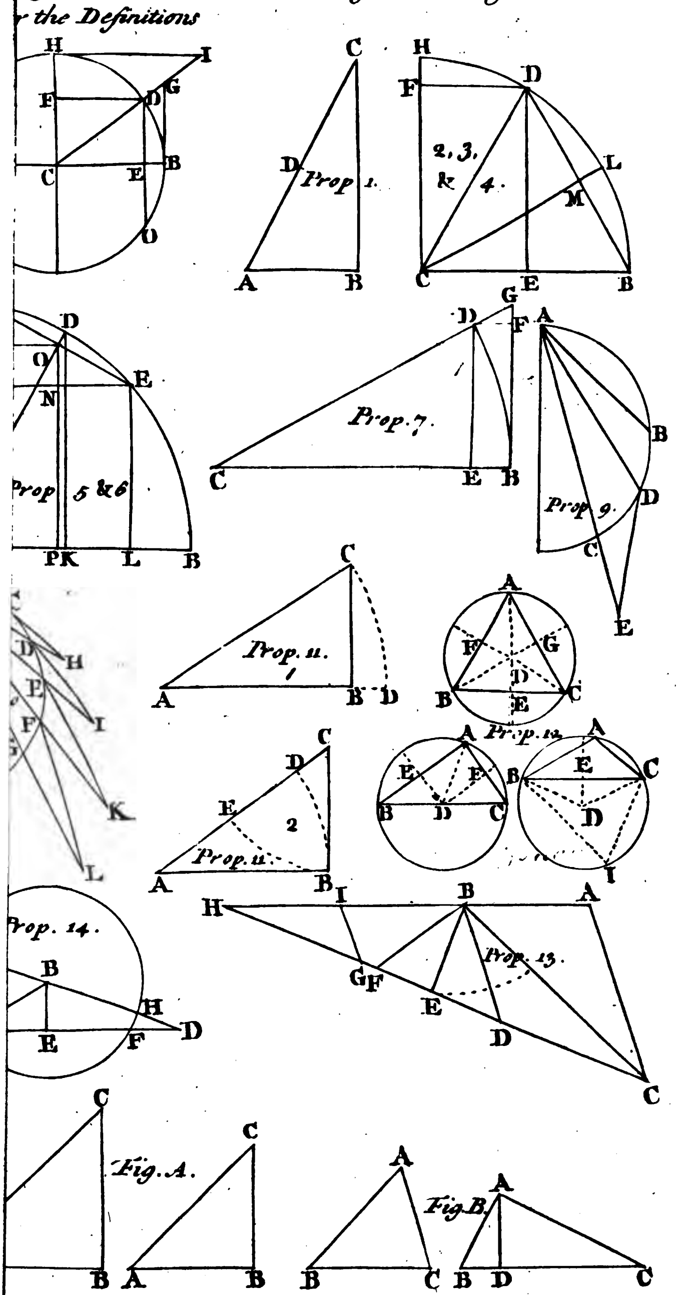
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	Given	Sought	Make, as
2	All the Angles A,B,C.	All the Sides A B, A C, B C.	S, C : S, A : : A B : B C. And S, C : S, B : A B : A C. Whence, if the Angles are given, the Proportions of the Sides may be found, but not the Sides themſelves, unleſs one of them be firſt known.
3	The two Sides A B, B C, and C, the Angle oppoſite to one of them.	The Angles A and B.	A B : B C : : S, C : S, A; which therefore may be found. When A B, the Side oppoſite to C, the given Angle, is longer than C B, the Side oppoſite to the ſought Angle, the ſought Angle is leſs than a Right one. But when it is ſhorter, becauſe the Sine of an Angle, and that of its Complement to two Right Angles, is the ſame, the Species of the Angle A muſt be firſt known, or the Solution will be ambiguous.
4	The two Sides A B, B C, and the interjacent Angle B.	The Angles A and C.	*Vid. Fig.* to *Prop.* 13. BC+AB : BC—AB : : T,$\frac{A+C}{2}$: T,$\frac{A-C}{2}$ Whence is known the Difference of the Angles A and C, whoſe Sum is given; and ſo (by *Prop.* 15.) the Angles themſelves will be given.
5	All the Sides A B, B C, A C,	The Angles.	*Vid. Fig.* B. Let the Perpendicular be drawn from the Vertex to the Baſe, and find the Segments of the Baſe by *Prop.* 14. *viz.* make as B C : A C+A B : : A C—A B : D C—D B. And ſo B D, D C, are given from this Analogy; and thence the Angles A B D, A C D, will be given by the Reſolution of Right-angled Triangles.

THE

ELEMENTS

OF

Spherical Trigonometry.

DEFINITIONS.

I. *THE Poles of a Sphere are two Points in the Superficies of the Sphere, that are the Extremes of the Axis.*

II. *The Pole of a Circle in a Sphere is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle are equal to one another.*

III. *A great Circle in a Sphere is that whoſe Plane paſſes thro' the Centre of the Sphere, and whoſe Centre is the ſame with that of the Sphere.*

IV. *A ſpherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

V. *A ſpherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the ſaid Circles.*

PROPOSITION I.

Great Circles A C B, A F B, *mutually biſect each other*.

FOR, ſince the Circles have the ſame Centre, their common Section ſhall be a Diameter of each Circle, and ſo will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being leſs than Semicircles, do not comprehend a Space; for they cannot, unleſs they meet each other in two oppoſite Points; that is, unleſs they are Semicircles.

PROPOSITION II.

If from the Pole C *of any Circle* A F B, *be drawn a Right Line* C D *to the Centre thereof, the ſaid Line will be perpendicular to the Plane of that Circle*, Vid. Fig. to Prop. 1.

LET there be drawn any Diameters E F, G H, in the Circle A F B; then, becauſe in the Triangles C D F, C D E, the Sides C D, D F, are equal to the Sides C D, D E, and the Baſe C F equal to the Baſe C E (by *Def.* 2.); then (*by* 8 *El.* 1.) ſhall the Angle C D F = Angle C D E, and ſo each of them will be a Right Angle. After the ſame manner we demonſtrate, that the Angles C D G, C D H, are Right Angles; and ſo (by 4 *El.* 11.) C D ſhall be perpendicular to the Plane of the Circle A F E. W. W. D.

Coroll. 1. A great Circle is diſtant from its Pole by the Interval of a Quadrant; for, ſince the Angles C D G, C D F, are Right Angles, the Meaſures of them, *viz.* the Arcs C G, C F, will be Quadrants.

2. Great Circles, that paſs thro' the Pole of ſome other Circle, make Right Angles with it; and, contrariwiſe, if great Circles make Right Angles with ſome other Circle, they ſhall paſs thro' the Poles of that other Circle; for they muſt neceſſarily paſs thro' the Right Line D C.

PROPOSITION III.

If a great Circle E C F *be deſcribed about the Pole* A; *then the Arc* C F, *intercepted between* A C, A F, *is the Meaſure of the Angle* C A F, *or* CDF. Vid. Fig. to Prop. 1.

THE Arcs A C, A F (*by Cor.* 1. *Prop.* 2.) are Quadrants; and, conſequently, the Angles A D C, A D F, are Right Angles: Wherefore (*by Def.* 6, *El.* 11.) the Angle C D F (whoſe Meaſure is the Arc C F) is equal to the Inclination of the Planes A C B, A F B, and alſo equal to the Spherical Angle C A F, or C B F. W. W. D.

Coroll. 1. If the Arcs A C, A F, are Quadrants, then ſhall A be the Pole of the Circle paſſing thro' the Points C and F; for A D is at Right Angles to the Plane F D C (*by* 4 *El.* 11.)

2. The vertical Angles are equal; for each of them is equal to the Inclination of the Circles; alſo the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles ſhall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehended by the two Sides alſo equal.

PROPOSITION V.

Alſo Triangles ſhall be equal and congruous, if one Side, together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually equilateral, are alſo mutually equiangular.

PROPOSITION VII.

In Iſoſceles Triangles, the Angles at the Baſe are equal.

PRO-

PROPOSITION VIII.

And if the Angles at the Baſe be equal, then the Triangle ſhall be Iſoſceles.

THESE five laſt Propoſitions are demonſtrated in the ſame manner, as in plane Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle is the ſhorteſt Way, between any two Points in the Superficies of the Sphere,

PROPOSITION X.

A Side of a Spherical Triangle is leſs than a Semicircle.

LET AC, AB, the Sides of the Triangles ABC, be produced till they meet in D: Then ſhall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPOSITION XI.

The three Sides of a Spherical Triangle are leſs than a whole Circle.

FOR BD+DC is greater than BC (*by Prop.* 9.); and, adding on each Side BA+AC; then DBA +DCA, that is a whole Circle, will be greater than BA+BC+AC, which are the three Sides of the Spherical Triangle ABC.

PROPOSITION XII.

In a Spherical Triangle ABC, *the greater Angle* A *is ſubtended by the greater Side.*

MAKE the Angle BAD=Angle B; then ſhall AD=BD (*by* 8 *of this*); and ſo BDC=DA+DC, and theſe Arcs are greater than AC. Wherefore the Side BC, that ſubtends the Angle BAC, is greater than the Side AC, that ſubtends the Angle B.

PROPOSITION XIII.

In any Spherical Triangle A B C, *if the Sum of the Legs* A B *and* B C *be greater, equal, or leſs, than a Semicircle, then the internal Angle at the Baſe* B A C *ſhall be greater, equal, or leſs, than the external and oppoſite Angle* B C D; *and ſo the Sum of the Angles* A *and* A C B *ſhall alſo be greater, equal, or leſs, than two Right Angles.*

FIRST, let $AB+AC=$ Semicircle $=AD$; then $BC=BD$, and the Angles BCD and D equal (*by* 8 *of this*); and therefore the Angle BCD ſhall be $=$ Angle A.

Secondly, let $AB+BC$ be greater than ABD; then ſhall BC be greater than BD; and ſo the Angle D (that is, the Angle A, *by* 12 *of this*) ſhall be greater than the Angle BCD. In like manner we demonſtrate, if $AB+BC$ be together leſs than a Semicircle, that the Angle A will be leſs than the Angle BCD: And becauſe the Angles BCD and BCA are $=$ two Right Angles, if the Angle A be greater than the Angle BCD, then ſhall A and BCA be greater than two Right Angles; if the Angle $A=BCD$, then ſhall A and BCA be equal to two Right Angles; and if A be leſs than BCD, then will A and BCA be leſs than two Right Angles. W. W. D.

PROPOSITION XIV.

In any Spherical Triangle G H D, *the Poles of the Sides, being joined by great Circles, do conſtitute another Triangle* X M N, *which is the Supplement of the Triangle* G H D; viz. *the Sides* N X, X M, *and* N M, *ſhall be Supplements of the Arcs that are the Meaſures of the Angles* D, G, H, *to the Semicircles*; *and the Arcs that are the Meaſures of the Angles* M, X, N, *will be the Supplements of the Sides* G H, G D, *and* H D, *to Semicircles.*

FROM the Poles G, H, D, let the great Circles XCAM, TMNO, XKBN, be deſcribed;

then, becauſe G is the Pole of the Circle X C A M, we ſhall have G M=Quadrant (*Cor.* 1. *Prop.* 2.); and ſince H is the Pole of the Circle T M O, then will H M be alſo a Quadrant, and ſo (*by Cor.* 1. *Prop.* 3.) M ſhall be the Pole of the Circle G H. In like manner, becauſe D is the Pole of the Circle X B N, and H the Pole of the Circle T M N, the Arcs D N, H N, will be Quadrants; and ſo (by *Cor.* 2. *Prop.* 3.) N ſhall be the Pole of the Circle H D. And becauſe G X, D X, are Quadrants, X will be the Pole of the Circle G D. Theſe Things premiſed,

Becauſe N K=Quadrant, and X B=Quadrant (*by Cor.* 1. *Prop.* 2.); then will N K+X B, that is, N X +K B=two Quadrants, or a Semicircle; and ſo N X is the Supplement of the Arc K B, or of the Meaſure of the Angle H D G, to a Semicircle. In like manner, becauſe M C=Quadrant, and X A=Quadrant, then will M C+X A, that is, X M+A C=two Quadrants, or a Semicircle; and, conſequently, X M is the Supplement of the Arc A C, which is the Meaſure of the Angle H G D. Likewiſe, ſince M O, N T, are Quadrants, we ſhall have M O+N T= O T+N M=Semicircle: And therefore N M is the Supplement of the Arc O T, or of the Meaſure of the Angle G H D to a Semicircle., W. W. D.

Moreover, becauſe D K, H T, are Quadrants, D K +H T, or K T+H D, are equal to two Quadrants, or a Semicircle; therefore K T, or the Meaſure of the Angle X N M, is the Supplement of the Side H D to a Semicircle. After the ſame manner it is demonſtrated, that O C, the Meaſure of the Angle X M N, is the Supplement of the Side G H; and B A, the Meaſure of the Angle X, is the Supplement of the Side G D. W. W. D.

PROPOSITION XV.

Equiangular Spherical Triangles are alſo equilateral.

FOR their Supplementals (*by* 14 *of this*) are equilateral, and therefore equiangular alſo; and ſo themſelves are likewiſe equilateral (*by Part* 2. *Prop.* 14.)

PROPOSITION XVI.

The three Angles of a Spherical Triangle are greater than two Right Angles, and leſs than ſix.

FOR the three Meaſures of the Angles G, H, D, together with the three Sides of the Triangle X N M, make three Semicircles (*by* 14 *of this*); but the three Sides of the Triangle X N M are leſs than two Semicircles (*by* 11 *of this*). Whereſore the three Meaſures of the Angles G, H, D, are greater than a Semicircle; and ſo the Angles G, H, D, are greater than two Right Angles.

The ſecond Part of the Propoſition is manifeſt; for, in every Spherical Triangle, the external and internal Angles, together, only make ſix Right Angles: Whereſore the internal Angles are leſs than ſix Right Angles.

PROPOSITION XVII.

If from the Point R, *not being the Pole of the Circle* AFBE, *there fall the Arcs* R A, R B, R G, R V, *of great Circles to the Circumference of that Circle; then the greateſt of thoſe Arcs is* R A, *which paſſes thro' the Pole* C *thereof; and the Remainder of it is the leaſt; and thoſe that are more diſtant from the greateſt are leſs than thoſe which are nearer to it, and they make an obtuſe Angle with the former Circle* A F B, *on the Side next to the greateſt Arc.* Vid. Fig. to Prop. 1.

BEcauſe C is the Pole of the Circle A F B, then ſhall C D, and R S, which is parallel thereto, be perpendicular to the Plane A F B. And if S A, S G, S V, be drawn, then ſhall S A (*by* 7 *El.* 3.) be greater than S G, and S G greater than S V. Whence, in the Right-angled plane Triangles R S A, R S G, R S V, we ſhall have R S q + S A q, or R A q, greater than R S q + S G q, or R G q; and ſo R A will be greater than R G, and the Arc R A greater than the Arc R G. In like manner, R S q + S G q, or R G q, ſhall be greater than R S q+S V q, or R V q; and ſo R G ſhall be greater than R V, and the Arc R G greater than the Arc R V.

2*dly*, The Angle R G A is greater than the Angle C G A, which is a Right Angle (*by Cor. Prop.* 3.); and the Angle R V A is greater than the Angle C V A, which alſo is a Right Angle. Therefore the Angles R G A, R V A, are obtuſe Angles.

PROPOSITION XVIII.

In Spherical Triangles A G R, A G X, *Right-angled at* A, *the Legs containing the Right Angle are of the ſame Affection with the oppoſite Angles; that is, if the Legs be greater or leſs than Quadrants, then, accordingly, will the Angles oppoſite to them be greater or leſs than Right Angles.* Vid. Fig. to Prop. 1.

FOR if A C be a Quadrant, then will C be the Pole of the Circle A F B; and the Angles A G C, A V C, will be Right Angles. If the Leg A R be greater than a Quadrant, then ſhall the Angle A G R be greater than a Right Angle (*by* 17 *of this*); and if the Leg A X be leſs than a Quadrant, the Angle A G X ſhall be leſs than a Right Angle.

PROPOSITION XIX.

If two Legs of a Right-angled Spherical Triangle be of the ſame Affection (and conſequently the Angles), that is, if they are both leſs, or both greater, than a Quadrant, then will the Hypothenuſe be leſs than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle A R V, or B R V, let F be the Pole of the Leg A R; then will R F be a Quadrant, which is greater than R V (*by* 17 *of this*).

PROPOSITION XX.

If they be of a different Affection, then ſhall the Hypothenuſe be greater than a Quadrant. Vid. Fig. to Prop. 1.

FOR, in the Triangle A R G, the Hypothenuſe R G is greater than R F, which is a Quadrant.

PROPOSITION XXI.

If the Hypothenuſe be greater than a Quadrant, then the Legs of the Right Angle, and ſo the Angles oppoſite to them, are of a different Affection; but if leſſer, of the ſame Affection. Vid. Fig. to Prop. 1.

THIS Propoſition, being the Converſe of the former ones, eaſily follows from them.

PROPOSITION XXII.

In any ſpherical Triangle A B C, *if the Angles at the Baſe* B *and* C *be of the ſame Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.*

IN the firſt Caſe, if the Perpendicular does not fall within, let it fall without the Triangle (as in *Fig.* 2.); then, in the Triangle A B P, the Side A P is of the ſame Affection with the Angle B. And, in like manner, in the Triangle A C P, A P is of the ſame Affection with the Angle A C P; therefore, ſince A B C and A C P are of the ſame Affection, the Angles A B C, A C B, ſhall be of a different Affection; which is contrary to the Hypotheſis.

In the ſecond Caſe, if the Perpendicular does not fall without, let it fall within (as in *Fig.* 1.) Then, in the Triangle A B P, the Angle B is of the ſame Affection with the Leg A P. So, likewiſe, in the Triangle A C P, the Angle C is of the ſame Affection with A P; and therefore the Angles B and C are of the ſame Affection; which is contrary to the Hypotheſis.

PROPOSITION XXIII.

In Spherical Triangles B A C, B H E, *Right-angled at* A *and* H, *if the ſame acute Angle* B *be at the Baſe* B A, *or* B H, *then the Sines of the Hypothenuſes ſhall be proportional to the Sines of the perpendicular Arcs.*

FOR the Right Lines C D, E F, being perpendicular to the ſame Plane, are parallel. Alſo, E B, D B, DP, perpendicular to the Radius OB, are likewiſe parallel: Wherefore the Planes of the Triangles EFR, CDP, are alſo parallel (*by* 15 *El.* 11.) Wherefore CP, ER, the common Sections of thoſe Planes, with the Plane paſſing thro' BE, CO, will be parallel (*by* 16 *El.* 11.) Therefore the Triangles CDP, EFR, ſhall be equiangular: Wherefore CP, the Sine of the Hypothenuſe BC, is to CD, the Sine of the perpendicular Arc CA, as ER, the Sine of the Hypothenuſe BE, to EF, the Sine of the perpendicular Arc EH. W. W. D.

PROPOSITION XXIV.

The ſame Things being ſuppoſed, AQ, HK, *the Sines of the Baſes, are proportional to* IA, GH, *the Tangents of the perpendicular Arcs.*

FOR, after the ſame manner as in the laſt Propoſition, we demonſtrate, that the Triangles QAI, KHG, are equiangular; whence, QA : AI :: KH : HG.

PROPOSITION XXV.

In a Spherical Triangle ABC, *Right-angled at* A, *as the Coſine of the Angle* B, *at the Baſe* BA, *is to the Sine of the vertical Angle* ACB, *ſo is the Coſine of the Perpendicular to the Radius.*

PREPARATION.

LET the Sides AB, BC, CA, be produced, ſo that BE, BF, CI, CH, be Quadrants; and from the Poles B and C draw the great Circles EFDG, IHG; then will the Angles at E, F, I, H, be Right Angles; and ſo D is the Pole of BAE (*by Cor.* 2. *Prop.* 2. *of this*), and G the Pole of IFCB: Alſo, AE will be = Complement of the Arc BA, and FE the Meaſure of the Angle B = GD, and DF their Complement: Alſo, BC ſhall be = FI = Meaſure of the Angle G, and CF their Complement: Likewiſe, CA = HD, and DC their Complement. Theſe Things premiſed, in the Triangles HIC, DCF, Right-angled at I and F, and having the ſame acute Angle C; ſince BA is leſs than a Quadrant, it will be, as S, DF : S, HI : : S, DC : S, HC; that is, the Coſine of the Angle B is to the Sine of the vertical Angle BCA, as the Coſine of CA is to Radius. W W D.

PROPOSITION XXVI.

The Coſine of the Baſe : *Coſine of the Hypothenuſe* : : R : *Coſ. of the Perpendicular.*

FOR in the Triangles AED, CFD, Right-angled at E, F, having the ſame acute Angle D; becauſe A E is leſs than a Quadrant, we have, S, E A : S, CF : : S, DA : S, DC. W. W. D.

PROPOSITION XXVII.

S, *of the Baſe* : R : : T, *of the Perpendicular* : T, *of the Angle at the Baſe.*

FOR in the Triangles BAC, BEF, Right-angled at A and E, and having the ſame acute Angle B; becauſe A C is leſs than a Quadrant, we have S, BA : S, BE : : T, AC : T, EF. W. W. D.

PROPOSITION. XXVIII.

Coſ. of the Vertical Angle : R : : T, *of the Perpendicular* : T, *of the Hypothenuſe.*

IN the Triangles GIF, GHD, Right-angled at I and H, and having the ſame acute Angle G, becauſe HD is leſs than HC, or a Quadrant, it is, as S, GH : S, GI : : T, HD : T, IF.

PROPOSITION XXIX.

S, *of the Hypothenuſe* : R : : S, *of the Perpendicular* : S, *of the Angle at the Baſe.*

IN the aforeſaid Triangles we have S, IF : S, GF : : S, HD : S, GD.

PRO-

PROPOSITION XXX.

R : *Coſ. of the Hypothenuſe* : : T, *of the vertical Angle* : *Cot. of the Angle at the Baſe.*

IN the Triangles HIC, DFC, Right-angled at I and F, and having the ſame acute Angle C, becauſe DF is leſs than a Quadrant, we have S, CI : S, CF : : T, HI : T, DF; that is, R : Coſ. BC : : T, C : Cot. B.

The laſt ſix Propoſitions are ſufficient for ſolving all the ſixteen Caſes of Right-angled Spherical Triangles. Theſe ſixteen Caſes, with their Analogies deduced from the ſaid Propoſitions, are as follow :

	Given beſides the Right Angle	Sought	*Vid. Fig.* to *Prop.* 25, 26, 27, 28, 29, 30.	
1	AC and C.	B.	R : Coſ. CA : : S, C : Coſ. B, of the ſame Kind with CA.	By the Inverſe of Prop. 25.
2	AC and B.	C.	Coſ. CA : R : : Coſ. B : S, C; this is ambiguous.	By Prop. 25.
3	B and C.	AC.	S, C : Coſ. B : : R : Coſ. CA, of the ſame Kind with the Angle B.	By Prop. 25, and 18.
4	AB, CA.	BC.	R : Coſ. BA : : Coſ. AC : Coſ. BC. If BA, AC, be of the ſame Affection, and not Quadrants, then BC will be leſs than a Quadrant. If they be of a different Affection, BC ſhall be greater than a Quadrant.	By Prop. 26, 19, and 20.

	Given beſides the Right Angle	Sought		
5	BA, BC.	AC.	Coſ. BA : R : : Coſ. BC : Coſ. CA. If BC be leſs than a Quadrant, then ſhall BA and CA be of the ſame Affection; if greater, of a different: But BA is given, and therefore the Species thereof. Wherefore the Species of AC is alſo given.	By Prop. 26, and 21.
6	BA, CA.	B.	S, BA : R : : T, CA : T, B, of the ſame Affection with the oppoſite Side CA.	By Prop. 27, and 18.
7	BA, B.	AC.	R : S, BA : : T, B : T, AC, of the ſame Kind with B.	By Prop. 27, and 18.
8	AC, B.	BA.	T, B : T, CA : : R : S, AB, ambiguous.	By Prop. 27.
9	BC, C.	AC.	R : Co. C : : T, BC : T, CA. If BC be leſs than a Quadrant, the Angles C and B are of the ſame Affection; if greater, of a different. Therefore, if the Species of the Angle B be given, then will AC be given.	By Prop. 28, and 21.
10	AC, C.	BC.	Coſ. C : R : : T, CA : T, BC. And ſo, if the Angle C and CA, be of the ſame Affection, then BC ſhall be leſſer than a Quadrant; if of a different, greater.	By Prop. 28, 20, 21.
11	BC, AC.	C.	T, BC : T, CA : : R : Coſ. C. If BC be leſs than a Quadrant, then CA and BA, and conſequently the Angles, ſhall be of the ſame Affection; if greater, of a different. But the Species of CA is given; therefore the Species of the Angle C will be alſo given.	By Prop. 28, and 21.

	Given beſides the Right Angle	Sought		
12	BC, B.	AC.	R : S, BC : : S, B : S, AC, of the ſame Species with B.	By Prop. 29, and 18.
13	AC, B.	BC.	S, B : S, AC : : R : S, BC, ambiguous.	By Prop. 29.
14	BC, AC.	B.	S, BC : R : : S, AC : S, B, of the ſame Species with CA.	By Prop. 29.
15	B, C.	BC.	T, C : Cot. B : : R : Coſ. BC. And ſo, if the Angles B and C are of the ſame Affection, then ſhall BC be leſs than a Quadrant, if of a different, greater.	By Prop. 30, 19, and 20.
16	BC, C.	B.	R : Coſ. BC : : T, C : Cot. B. And ſo, if BC be leſs than a Quadrant, the Angles C and B ſhall be of the ſame Affection; if greater, of a different. But the Species of the Angle C is given; therefore the Species of the Angle B will be given alſo.	By Prop. 30, and 31.

Of the Solution of Right-angled Spherical Triangles, by the five circular Parts.

THE Lord *Neper* (the noble Inventor of Logarithms) by a due Conſideration of the Analogies by which Right-angled Spherical Triangles are ſolved, found out two Rules, eaſy to be remembered, by means of which, all the ſixteen Caſes may be ſolved: For ſince, in theſe Triangles, beſides the Right Angles, there are three Sides, and two Angles; the two Sides comprehending the Right Angle, and the Complements of the Hypothenuſe, and the two other Angles, were called by *Neper*, *Circular Parts*; and when there are given any two of the ſaid Parts, and a third is ſought; one of theſe three, which is called the *Middle Part*, either lies between the two other Parts, which are called *Adjacent Extremes*; or is ſeparated from them, and then are called *Oppoſite Extremes*: So if the Complement of the Angle B (*Fig.* to *Prop.* 25.) be ſuppoſed the middle Part, then the Leg A B, and the Complement of the Hypothenuſe B C, are adjacent extreme Parts; but the Complement of the Angle C, and the Sides A C, are oppoſite Extremes. Alſo, if the Complement of the Hypothenuſe B C be ſuppoſed the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs A B, A C, are oppoſite Extremes. In like manner, ſuppoſing the Leg A B the middle Part, the Complement of the Angle B and A C are adjacent Extremes; for the Right Angle A doth not interrupt the Adjacence, becauſe it is not a circular Part. But the Complement of the Angle C, and the Complement of the Hypothenuſe B C, are oppoſite Extremes to the ſaid middle Part. Theſe Things premiſed,

RULE I.

In any Right-angled Spherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

RULE

RULE II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Coſine of the oppoſite Parts.

Each of the Rules have three Caſes: For the middle Part may be the Complement of the Angle B, or C; or the Complement of the Hypothenuſe B C; or one of the Legs, A B, A C.

Caſe 1. Let the Complement of the Angle C be the middle Part; then ſhall A C, and the Complement of the Hypothenuſe B C, be adjacent Extremes. By *Prop.* 28. the Coſine of the vertical Angle C is to Radius as the Tangent of C A is to the Tangent of the Hypothenuſe B C. Then (by Alternation) we ſhall have Coſ. C : T, C A : : R : T, B C. But R : T, B C : : Cot. B C : R (as has been before ſhewn). Wherefore Coſ. C : T, A C : : Cot. B C : R; whence R Coſ. C = T, A C × Cot. B C.

And the Complement of the Angle B, and A B, are oppoſite Extremes to the ſame middle Part, the Complement of the Angle C; and (*by Prop.* 25.) as the Coſine of the Angle C, to the Sine of the Angle C D F, ſo is the Coſine of D F to Radius. But the Sine of C D F = A E = Coſ. B A, and Coſ. D F = S, E F = S, Angle B. Whence it will be, as Coſ. C : Coſ. B A : : S, B : R. And R × Coſ. C = Coſ. B A × S, B; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Coſines of the oppoſite Extremes.

Caſe 2. Let the Complement of the Hypothenuſe B C be the middle Part; then the Complements of the Angles B and C will be adjacent Extremes. In the Triangle D C F (*by Prop.* 27.) it is, as S, C F : R : : T, D F : T, C. Whence (by Alternation) S, C F : T, D F : : (R : T, C : :) Cot. C : R. But S, C F = Coſ. B C and T, D F = Cot. B. Wherefore R × Coſ. B C = Cot. C × Cot. B; that is, Radius drawn into the Sine of the middle Part is equal to the Product of the Tangents of the adjacent extreme Parts.

And B A, A C, are the oppoſite Extremes to the ſaid middle Part, *viz.* the Complement of B C; and (*by Prop.* 26.) Coſ. B A : Coſ. B C : : R : Coſ. A C. Wherefore we ſhall have R×Coſ. B C=Coſ. B A Coſ. A C.

Caſe 3. Laſtly, let A B be the middle Part; and then the Complement of the Angle B, and A C, will be adjacent Extremes, and (*by Prop.* 27.) S, A B : R : : T, C A : T, B. Whence S, A B : T, C A : : (R : T, B : :) Cot. B : R. And ſo R×S, A B=T, C A×Cot. B.

Moreover, the Complements of B C, and the Angle C, are oppoſite Extremes to the ſame middle Part A B; and in the Triangle G H D (*by Prop.* 25.) we have Coſ. D : S, D G H : : Coſ. G H : R. But Coſ. D=Coſ. A E=S, A B, and S, G=S, I F=S, B C. Alſo, Coſ. G H=S, H I=S, C. Wherefore it will be, as S, A B : S, B C : : S, C : R. And hence R×S, A B=S, B C×S, C.

And ſo, in every Caſe, the Rectangle under the Radius, and the Sine of the middle Part, ſhall be equal to the Rectangle under the Coſines of the oppoſite Extremes, and to the Rectangle under the Tangents of adjacent Extremes: And, conſequently, if the aforeſaid Equations be reſolved into Analogies (*by* 16 *El.* 6.), the unknown Parts may be found by the Rule of Proportion. And if that Part ſought be the middle one, then ſhall the firſt Term of the Analogy be Radius, and the ſecond and third, the Tangents or Coſines of the extreme Parts. If one of the Extremes be ſought, the Analogy muſt begin with the other; and the Radius, and the Sine of the middle Part, muſt be put in the middle Places, that ſo the Part ſought may be in the fourth Place.

IN Oblique-angled Spherical Triangles (*Fig. to Prop.* 31.) B C D, if a perpendicular Arc A C be let fall from the Angle C to the Baſe, continued, if need be, ſo as to make two Right-angled Spherical Triangles B A C, D A C; then, by thoſe Right-angled Triangles, may moſt of the Caſes of Oblique-angled ones be ſolved.

PROPOSITION XXXI.

The Coſines of the Angles B *and* D, *at the Baſe* B D, *are proportional to the Sines of the vertical Angles* B C A, D C A.

FOR Coſ. Angle B : S, BCA : : (Coſ. CA : R : :) Coſ. D : S, DCA (*by* 25 *of this.*)

PROPOSITION XXXII.

The Coſines of the Sides B C, D C, *are proportional to the Coſines of the Baſes* B A, D A.

FOR Coſ. BC : Coſ. BA : : (Coſ. CA : R : :) Coſ. DC : Coſ. DA (*by* 26 *of this.*)

PROPOSITION XXXIII.

The Sines of the Baſes B A, D A, *are in a reciprocal Proportion of the Tangents of the Angles* B *and* D, *at the Baſe* B D.

BEcauſe (*by* 27 *of this*) S, B A : R : : T, AC : T, of the Angle B. And by the ſame inverſly, R : S, D A : : T, of the Angle D : T, A C. Then will it be (by the Equality of perturbate Ratio, according to *Prop*. 23. *El*. 5.) S, B A : S, D A : : T, Angle D : T, Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides B C, D C, *are in a reciprocal Proportion of the Coſines of the vertical Angles* B C A, D C A.

BEcauſe, by alternating the 28th *Propoſition*, we have T, BC : R : : T, C A : Coſ. B C A, and by the ſame, R : Coſ. D C A : : T, D C : T, C A. Wherefore, by Equality of perturbate Proportion, T, BC : Coſ. DCA : : T, DC : Coſ. BCA.

PROPOSITION XXXV.

The Sines of the Sides B C, D C, *are proportional to the Sines of the oppoſite Angles* D *and* B.

BEcauſe (*by the 29th of this*) S, B C : R : : S, C A : S, of the Angle B; and, by the ſame, inverting, R : S, D C : : S, Angle D : S, of C A. Whence, by Equality of perturbate Ratio, S, B C : S, D C : : S, D : S, B.

PROPOSITION XXXVI.

In any Spherical Triangle A B C, *the Rectangle* C F × A E, *or* F M × A E, *contained under the Sines of the Legs* B C, B A, *is to the Square of the Radius, as* I L *or* I A—L A *the Difference of the verſed Sines of the Baſe* C A, *and the Difference of the Legs* A M, *to* G N, *to the verſed Sine of the Angle* B.

LET a great Circle P N be deſcribed from the Pole B; and let BP, BN, be Quadrants; and then P N is the Meaſure of the Angle B; alſo, deſcribe from the ſame Pole B a leſſer Circle C F M thro' C; the Planes of theſe Circles ſhall be perpendicular to the Plane B O N (*by the 2d of this.*) And P G, C H, being perpendicular in the ſame Plane, fall on the common Sections O N, F M; ſuppoſe in G, H. Again, draw H I perpendicular to A O; and then the Plane drawn thro' C H, H I, ſhall be perpendicular to the Plane A O B. Whence A I, which is perpendicular to H I, will be perpendicular to the Right Line C I (*by Def.* 4. *El.* 11.); and ſo A I is the verſed Sine of the Arc A C; and A L the verſed Sine of the Arc A M =B M—B A=B C—B A. The Iſoſceles Triangles C F M, P O N, are equiangular, ſince M F, N O, as alſo C F, P O (*by* 16 *El.* 11.) are parallel. Wherefore, if Perpendiculars C H, P G, be drawn to the Sides F M, O N, the Triangles will be divided ſimilarly, and we ſhall have F M : O N : : M H : G N. Alſo, becauſe the Triangles A O E, D I H, D L M, are equiangular, we ſhall have AE : AO : : IL : MH.

But it has been proved, that FM : ON : : MH : GN. Wherefore it ſhall be, as $AE \times FM : AO \times ON : : IL \times MH : MH \times GN$, or ſo is IL to GN; that is, the Rectangle under the Sines of the Legs, is to the Square of Radius, as the Difference of the verſed Sines of the Baſe, and the Difference of the Legs BC, BA, is to the verſed Sine of the Angle B. W. W. D.

PROPOSITION XXXVII.

The Difference of the verſed Sines of two Arcs, drawn into half the Radius, is equal to the Rectangle under the Sine of half the Sum, and the Sine of half the Difference, of thoſe Arcs.

LET there be two Arcs, BE, BF, whoſe Difference EF let be biſected in D; then ſhall BD be the half Sum, and FD the half Difference of thoſe Arcs. $GE = IL$ is the Difference of the verſed Sines of the Arcs BE, BF; alſo, FO is the Sine of the half Difference of the Arcs. And becauſe the Triangles CDK, FEG, are equiangular, we have $DK : GE : : (CD : FE : :) \frac{1}{2}CD : \frac{1}{2}FE$. Whence $DK \times \frac{1}{2}FE$, or $DK \times FO = GE \times \frac{1}{2}CD = IL \times \frac{1}{2}CD$. W. W. D.

PROPOSITION XXXVIII.

The verſed Sine of any Arc, drawn into half the Radius, is equal to the Square of the Sine of one half of the ſaid Arc.

THE Triangles CBM, DEB, are equiangular, ſince the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore EB : BD : : BM, : BC. And then will $EB \times BC = BM \times BD$; and $EB \times \frac{1}{2}BC = BM \times \frac{1}{2}BD = BMq$. W, W, D.

PROPOSITION XXXIX.

In any Spherical Triangle A B C, *whoſe Legs, containing the Angle* B, *are* B C, A B, *and Baſe ſubtending that Angle* A C; *if the Arc* A M *be taken = Difference of the Legs* = B C—A B; *then ſhall the Rectangle under the Sines of the Legs* B C, B A, *be to the Square of the Radius, as the Rectangle, under the Sine of the Arc* $\frac{AC+AM}{2}$ *and the Sine of the Arc* $\frac{AC-AM}{2}$, *is to the Square of the Sine of one half of the Angle* B. Vid. Fig. to Prop. 36.

BEcauſe the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as I L is to the verſed Sine of the Angle B, or as $\frac{1}{2}R\times IL$ to $\frac{1}{2}R$ drawn into the verſed Sine of the Angle B (*by Prop.* 36. *of this.*) And ſince $\frac{1}{2}R\times IL$ = Rectangle, under the Sines of the Arcs $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$, (*by Prop.* 37. *of this.*) And alſo $\frac{1}{2}R$ drawn into the verſed Sine of the Angle B is equal to the Square of the Sine of one half of the Angle B (*by Prop.* 38 *of this.*) Therefore the Rectangle under the Sines of the Sides, to the Square of Radius, ſhall be as the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$ is to the Square of the Sine of one half the Angle B. W. W. D.

The twelve Caſes of oblique-angled Spherical Triangles are as follow:

	Given	Sought	Make, as	
1	Angles B, D, and B C.	Angle C.	R : Coſ. B C : : T,B : Cot. BCA (*by Prop.* 30. *of this*): Alſo Coſ. B : S, DCA : : Coſ. D : S, DCA (*by* 31. *of this*). Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = B C D. Whether the Perpendicular falls within, or without the Triangle, may be known from the Affection of the Angles B and D (*by* 22. *of this*); which Admonition ought to be obſerved in the following Solutions.	*In the Original the Proportion was thus;* Coſ. B C : R : : T, B : Cot. B C A.
2	Angles B, BCD and the Side BC.	Angle D.	R : Coſ. BC : : T,B : Cot. BCA (*Prop.* 30 *of this.*) And S, BCA : S, DCA : : Coſ. B : Coſ. D (*by Prop.* 31.) If B C A be leſs than BCD, the Angle D ſhall be of the ſome Affection with the Angle B. If B C A be greater than the Angle BCD, then the Angles B and D ſhall be of a different Affection, by the Converſe of *Prop.* 22.	*This Proportion in the Original was as in the foregoing. The Species of the Angle* B C A *may be known by* Prop. 18, *and* 19.
3	The Sides B D, C D, and the Angle B.	The Side B D.	R : Coſ. B : : T, BC : T, BA (*by* 28. *of this.*) And Coſ. B C : Coſ. B A : : Coſ. D C : Coſ. C A (*by* 32. *of this.*) The Sum or Difference of B A and D A, according as the Perpendicular falls within or without the Triangle, is equal to B D; which cannot be known, unleſs the Species of the Angle D be firſt known.	

Given

	Given	Sought	Make, as
4	The Sides B C, D B, and the Angle B.	The Side C D.	R : Coſ. B : : T, BC : T, B A, (*by* 28. *of this.*) And Coſ. B A : Coſ. BC : : Coſ. DA : Boſ. D C (*by Prop.* 32. *of this.*) According as DA is ſimilar or diſſimilar to CA, or to the Angle BDC, ſo ſhall DC be leſſer or greater than a Quadrant (*by* 19, *and* 20. *of this.*)
5	Angles B, D, and the Side B C.	The Side B D.	R : Coſ. B : : T, BC : T, BA (*by* 28: *of this.*) And T, D, : T, B : : S, BA : S, D A (*by* 33. *of this.*) The Sum or Difference of B A and D A is = B D.
6	The Sides B C, B D, and the Angle B.	Angle D.	R : Coſ. B : : T, BC : T, BA (*by Prop.* 28. *of this.*) And S, D A : S, BA : : T, B : T, D (*by* 33. *of this.*) According as B D is greater or leſſer than B A, the Angle D ſhall be ſimilar or diſſimilar to the Angle B (*by* 22. *of this.*)
7	The Sides B C, D C, and the Angle B.	Angle C.	Coſ. BC : R : : Cot. B : T, BCA (*by* 30. *of this.*) And T, DC : T, B C : : Coſ. BCA : Coſ. D C A (*by* 34. *of this.*) The Sum or Difference of the Angles B C A, D C A, according as the Perpendicular falls within or without the Triangle, is equal to the Angle B C D.
8	The Angles B C D, and B, and the Side B C.	The Side D C.	Coſ. BC : R : : Cot. B: T, BCA (*by* 30. *of this.*) Alſo, Coſ. DCA : Coſ. BCA : : T, BC : T, DC (*by* 34. *of this.*) If the Angle DCA be ſimilar to the Angle B (that is, if AD be ſimilar to CA), then DC ſhall be leſs than a Quadrant. If the Angles DCA and B be diſſimilar, then DC ſhall be greater than a Quadrant, which follows (*from Prop.* 18, 19, *and* 20. *of this.*)

	Given	Sought	Make, as
9	The Sides B C, D C, and the Angle B.	The Angle D.	S, C D : S, B : : S, B C : S, D; which is ambiguous. The Analogy follows from *Prop.* 35. of this.
10	The Angles B, D, and the Side B C.	The Side D C.	S, D : S, B C : : S, B : S, D C; which Side is ambiguous.
11	All the Sides A B, B C. C A. *Vid. Fig. Prop.* 36.	The Angle B.	As the Rectangle under the Sines of the Legs A B, B C : the Square of Radius : : the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B (*by Pro.* 39.)
12	All the Angles G, H, D. *Vid. Fig. Prop.* 14.	The Side G D.	In the Triangle X N M, the Arc M N is the Complement of the Angle G H D to a Semicircle. X M is the Complement of the Angle G, and X N the Complement of the Angle D : And the Angle X, the Complement of the Side G D to a Semicircle. Wherefore, if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the ſame as in *Caſe* 11. of this; ſince Arcs, and their Complements to Semicircles, have the ſame Sines.

The

The following REMARK, *by* SAMUEL CUNN.

THAT this is true but in a particular Caſe, *viz.* when two of the Angles of the Triangle are Right ones; and two of the Sides Quadrants, may be thus demonſtrated: For, if poſſible, let ſome Triangle R S T, *Fig.* to *Prop.* 14th, be ſuch, that its Sides R S, S T, T R, be equal to the Meaſures of G H D, H G D, G D H, the Angles of a Triangle G H D; and, alſo, that the Meaſures of R S T, S T R, T R S, the Angles of the Triangles R S T, be equal to G H, G D, H D, the Sides of the Triangle G H D; and produce M X, M N, two Sides of the ſupplemental Triangle, to Semicircles, and they will meet ſomewhere, ſuppoſe at E; and there will be conſtructed thereby the Triangle N E X, of which X E (the Supplement of X M, which, by the 14th *Prop.* was the Supplement of the Meaſure of the Angle H G D) is equal to the Meaſure itſelf of the ſame Angle H G D: And, in like manner, N E, the Supplement of N M, which, by the 14th *Prop.* was the Supplement of the Meaſure of the Angle G H D, is equal to the Meaſure itſelf of the ſame Angle G H D. But the third Side X N is not the Meaſure of the third Angles G H D, but its Supplement, by the 14th *Prop.* Moreover, of the Angle E X N (whoſe Supplement is N X M), the Meaſure, by the 14th *Prop.* is equal to G D; and of the Angle X N E (whoſe Supplement is M N X) the Meaſure, by the 14th *Prop.* is equal to H D. But of the third N E X (which is equal to N M X) the Meaſure is not equal to G H, but its Supplement.

Now make N V = R T = B K, the Meaſure of the Angle G D H, and draw the great Circle E V. And ſince R S, by Suppoſition, is equal to the Meaſure of the Angle G H D, which is equal to E N; and ſince the Meaſure of the Angle S R T is, by Suppoſition, equal to D H, which is alſo equal to the Meaſure of the Angle X N E; the Angle X N E is equal to the Angle R. Then, conſequently, by the 4th *Prop.* the Triangles S R T, E N V, will have the Baſe S T equal to the Baſe

E V; the Angle T to the Angle N V E; and the Angle S, to the Angle N E V. But S T (which is equal to E V), by Suppoſition, is equal to the Meaſure of the Angle H G D; to which Meaſure X E is alſo equal: Therefore E V is equal to X E; and, conſequently, by the 7th *Prop.* the Angle E V X is equal to the Angle E X V; and the Angle E X V (whoſe Meaſure, as hath been ſhewn above, is equal to G D) is equal to the Angle T (or N V E), ſince, by Suppoſition, the Meaſure of this is alſo equal to G D. Therefore the Angle E V X is equal to the Angle E V N, and ſo both Right ones; and, conſequently, E X V a Right one alſo. Therefore, by the 2d *Cor.* to the 2d *Prop.* E V and E X are both Quadrants.

But if E V be a Quadrant, and at Right Angles to N X, then E, by 2d *Prop.* and its *Coroll.* is the Pole of N X; and ſo E N a Quadrant alſo, and the Angle E N V a Right one. Therefore, if the Sides of a Triangle (N E V, or its Equal) R S T are equal to the Meaſures of the Angles of ſome other Triangle G H D, and the Meaſures of the Angles of the former, equal to the Sides of the latter; two Sides of ſuch a Triangle R S T, or G H D, muſt be Quadrants, and two Angles of each Right ones.

Therefore, if a Triangle R S T be conſtructed, whoſe Sides are equal to the Meaſures of the Angles of another Triangle G H D; the Meaſures of the Angles of the Triangle R S T ſhall not be equal to the Sides of the Triangle G H D, unleſs in the one Caſe before-mentioned. Therefore the Meaſures of the Angles of the Triangle G H D, uſed as the Sides of a Triangle in the 11th *Caſe*, will not give us a Side of G H D, but the Meaſure of an Angle of the Triangle R S T, unleſs in the one afore-mentioned Caſe; *which was to be demonſtrated.*

But to find a Side G D of a Spherical Triangle G H D, whoſe Angles are all given, produce M N, that Side of the Supplemental Triangle, which is equal to the Supplement of the Meaſure of G H D, the Angle oppoſite to the Side ſought, and M X, either of the other Sides, till they meet, as in E. And there, as hath been before ſhewn, the Sides E X, E N, of the Triangle E X N, are exactly equal to the Meaſures of the Angles HGD, GHD, of the Triangle GHD; and of the Angles EXN, ENX, of the Triangle EXN, the Meaſures are equal to GD, HD. But the Side XN is equal to the Supplement of the Meaſure of the Angle GDH. And of the Angle XEN, the Meaſure is equal to the Supplement of GH.

Therefore the SOLUTION is thus:

Change one of the Angles GDH, adjacent to the Side ſought, into its Supplement; and then work with the Meaſures of the Angles as tho' they were Sides; and the Reſult will be GD, the Side ſought.

The preceding Fault, as well as the Omiſſions hereafter mentioned, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and many other Trigonometrical Writers.

In the Solution of our 9th and 10th Caſes, they have told us, that the *Quæſita* are ambiguous; which ſometimes, indeed, is true, but ſometimes alſo falſe. Therefore, as I conceive it, they ought to have laid down Rules, by Help of which we might diſcover when the *Quæſita* are ambiguous, and when not.

This Overſight may be corrected by the following Directions; wherein, becauſe every Sine correſponds to two Arcs, to one leſs than a Quadrant, and to another, which is the Supplement of the former to a Semicircle (a true Diſtinction of which of theſe are to uſed, being neceſſary to be known, before a proper Solution can be given to ſuch Problems as theſe are), I ſhall beg Leave for Brevity-ſake, to call the leſſer Arc the acute Value, and the greater the obtuſe; whether the Sine be of an Angle, or a Side.

In the tenth Caſe, there are given two Angles, B, *and* D, *and* BC, *a Side oppoſite to one of thoſe Angles* D, *to find* DC, *the Side oppoſite to the other.*

TO the acute Value of DC, and alſo to its obtuſe one, add BC; and if each of theſe Sums are

$\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles; both the Values of D C may be admitted, and then it is ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, only one Value of D C can be true, *viz.* the $\left\{\begin{matrix}\text{obtuſe}\\ \text{acute}\end{matrix}\right\}$ one; and then it is not ambiguous.

In the ninth Caſe, there are given two Sides B C, D C, *and one Angle* B, *oppoſite to* D C, *one of thoſe Sides, to find* D *the Angle oppoſite to the other.*

TO the acute Value of D, and alſo to its obtuſe Value, add B; and if each of theſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, when the Sum of the Sides is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, both the Values of D may be admitted, and conſequently D is ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, only one Value of D is true, *viz.* the $\left\{\begin{matrix}\text{obtuſe}\\ \text{acute}\end{matrix}\right\}$ one; and then it is not ambiguous.

Nor are we better uſed in the firſt Caſe; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles; yet, in each of thoſe Varieties, the *Quæſita* will be ſometimes ambiguous, and ſometimes not.

In the firſt Caſe there are given two Angles B, D, *and* B C, *a Side oppoſite to* D, *one of them, to find* C *the third Angle.*

1. Let the Perpendicular fall within; that is, let the given Angles be of the ſame Species.

TO the acute Value of D C A, and alſo to its obtuſe one, add the Angle B C A; and if each of theſe Sums is leſs than two Right Angles, then either the acute Value of D C A, or its obtuſe one added to B C A, gives a Value of B C D; which, therefore, is ambiguous. And when only one of theſe Sums is leſs than two Right Angles, the acute Value of D C A, added to B C A, gives the only Value of B C D; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of a different Species.

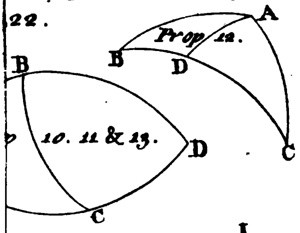
WHEN the obtuſe Value of the Angle D C A is leſs than the Angle B C A, the Angle B C D may be had by ſubtracting either Value of D C A from B C A; and then B C D is ambiguous. But when the obtuſe Value of D C A is not leſs than B C A, the acute Value of D C A, taken from B C A, gives the ſingle Value of B C D; which, therefore, is not ambiguous; tho' in both Varieties the Perpendicular fell without.

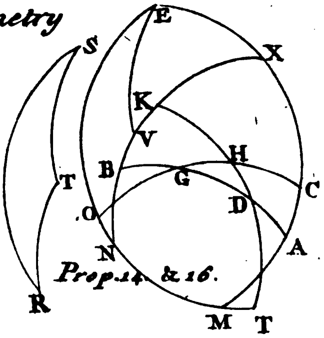
In the fifth Caſe we lie under the ſame Misfortune, where there are given, as in the firſt, the Angles B, D, *and the Side* B C, *to find* B D *the Side lying between thoſe given Angles.*

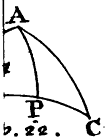
1. When the Perpendicular falls within; that is, when the given Angles are of the ſame Species.

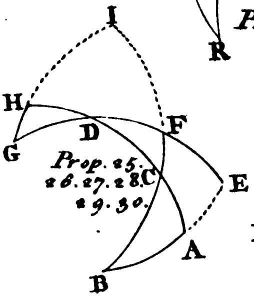
TO the acute Value of D A, and ſo alſo to its obtuſe one, add B A; and if each of theſe Sums is leſs than a Semicircle, then either the acute Value of D A, or its obtuſe one, added to B A, gives the Value of B D; which

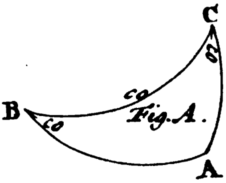
Spherical Trigonometry

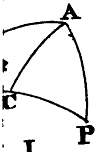


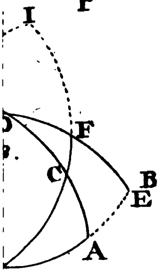


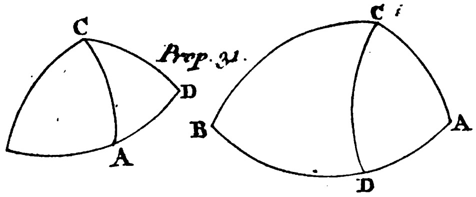


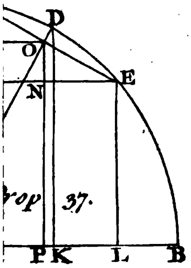


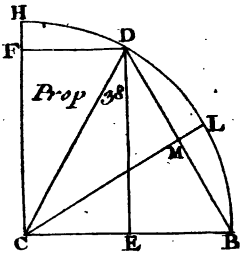












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which thence is ambiguous. And when only one of theſe Sums is leſs than a Semicircle, the acute Value of D A, added to B A, gives the only Value of B D; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the obtuſe Value of D A is leſs than B A, B D will be had by ſubtracting either Value of D A from B A; and then B D is ambiguous. But when the obtuſe Value of D A is not leſs than B A, the acute Value of D A, taken from B A, leaves the only Value of B D; which, therefore, is not ambiguous, tho' in both Varieties the Perpendicular fell without.

In the third, we have the ſame Omiſſion; where there are given two Sides B C, C D, *and* B *an Angle oppoſite to* C D *one of them, to find the third Side* B D.

FIRST, we may obſerve, that the Species of D A is always known; for it is of $\left\{\begin{array}{l}\text{the ſame}\\ \text{a different}\end{array}\right\}$ Affection with the Angle B, when D C is $\left\{\begin{array}{l}\text{leſs}\\ \text{greater}\end{array}\right\}$ than a Quadrant. And,

If A D be leſs than A B, and alſo the Sum of A D and A B leſs than a Semicircle; then A D, either added to, or ſubtracted from A B, will give the Value of B D; which, therefore, is ambiguous.

But if A D be not leſs than A B, or if their Sum be not leſs than a Semicircle; then their Sum in the former, and their Difference in the latter Variety, ſhall give one ſingle Value of B D; and then it is not ambiguous.

The ſeventh Caſe much reſembles the third; for there are given two Sides B C, C D, *and* B *an Angle, oppoſite to* C D *one of them; to find the Angle* B C D, *lying between thoſe two Sides.*

AND here we may obſerve, that the Species of the Angle D C A is known; for it is of $\left\{\begin{matrix}\text{the ſame}\\ \text{a different}\end{matrix}\right\}$ Kind with the Angle B, when D C is $\left\{\begin{matrix}\text{leſs}\\ \text{greater}\end{matrix}\right\}$ than a Quadrant. And,

If D C A be leſs than B C A, and the Sum of D C A and B C A leſs than two Right Angles; then D C A, either added to, or ſubtracted from, B C A, will give the Angle B C D; which, therefore, is ambiguous.

If D C A be not leſs than B C A, or the Sum of D C A and B C A not leſs than two Right Angles; then their Sum in the former, and their Difference in the latter, Variety, ſhall give the ſingle Value of B C D; which, then, is not ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side D C, or the Angle D, as taught in the Remarks on the 9th and 10th Caſes; they will find the ſeveral Varieties in the 1ſt, 3d, 5th, and 7th, to be as here laid down in theſe eaſy Rules.

The Truth of theſe Rules may be eaſily deduced from the 10th, 13th, 18th, and 22d *Prop. of this*; and the 2d, 8th, and 13th *Examples*, following *Prop.* 30. *of this*.

In our third *Caſe* of oblique plane Triangles, our Author ſhould have added this:

If A B be leſs than B C, the Angle A is ambiguous; otherwiſe, not.

A SHORT

TREATISE

OF THE

Nature *and* Arithmetic

OF

LOGARITHMS.

The PREFACE.

THE Mathematics formerly received conſiderable Advantages; firſt, by the Introduction of the Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it ſince reaped, at leaſt, as much from the Invention of Logarithms, as from both the other two. The Uſe of theſe, every one knows, is of the greateſt Extent, and runs through all Parts of Mathematics. By their Means it is that Numbers almoſt infinite, and ſuch as are otherwiſe impracticable, are managed with Eaſe and Expedition. By their Aſſiſtance the Mariner ſteers his Veſſel, the Geometrician inveſtigates the Nature of the higher Curves, the Aſtronomer determines the Places of the Stars, the Philoſopher accounts for* *other Phænomena of Nature; and, laſtly, the Uſurer computes the Intereſt of his Money.*

The Subject of the following Treatiſe has been cultivated by Mathematicians of the firſt Rank; ſome of whom, taking in the whole Doctrine, have indeed written learnedly, but ſcarcely intelligibly to any but Maſters. Others, again, accommodating themſelves to the Apprehenſion of Novices, have ſelected out ſome of the moſt eaſy and obvious Properties of Logarithms, but have left their Nature, and more intimate Properties, untouched. My Deſign therefore in the following Tract is, to ſupply what ſeemed ſtill wanting, viz. *to diſcover and explain the Doctrine of Logarithms, to thoſe who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the firſt that conſtructed and publiſhed a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciouſly received by all Mathematicians, who were immediately ſenſible of the extreme Uſefulneſs thereof. And tho' it is uſual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is univerſally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The ſame Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr.* Henry Briggs, *Profeſſor of* Geometry *at* Oxford, *who was hereby introduced as a Sharer in the compleating thereof: But the Lord* Neper *dying, the whole Buſineſs remaining was devolved upon Mr.* Briggs, *who with prodigious Application, and an uncommon Dex-*

Dexterity, compoſed a Logarithmic Canon, agreeable to that new Form, for the firſt twenty Chiliads of Numbers (*or from* 1 *to* 20000), *and for eleven other Chiliads*, viz, *from* 90000 *to* 1010000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publiſhed at* London *in the Year* 1624.

Adrian Vlacq *publiſhed again this Canon at* Gouda *in* Holland, *in the Year* 1628, *with the intermediate Chiliads, before omitted, filled up according to* Briggs's *Preſcriptions; but theſe Tables are not ſo uſeful as* Briggs's, *becauſe the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *has alſo calculated the Logarithms of the Sines and Tangents of every Degree, and the Hundredth Parts of Degrees, to* 15 *Places of Figures; and has ſubjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called artificial Sines and Tangents. Theſe Tables, together with their Conſtruction and Uſe, were publiſh'd after* Briggs's *Death, at* London, *in the Year* 1633, *by* Henry Gillebrand, *and by him called* Trigonometria Britannica.

Since then, there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

The beſt Diſpoſition of theſe Tables, in my Opinion, is that firſt thought of by Nathanael Roe, *of* Suffolk; *and, with ſome Alterations for the* *better, followed by* Sherwin *in his Mathematical Tables, publiſhed at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000, *conſiſting of ſeven Places of Figures. To which are ſubjoined the Differences, and proportional Parts, by means of which, may be found eaſily the Logarithms of Numbers to* 10000000; *obſerving, at the ſame Time, that theſe Logarithms conſiſt only of ſeven Places of Figures. Here are alſo the Sines, Tangents, and Secants, with their Logarithm, and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Uſe in practical Mathematics.*

OF THE
Nature *and* Arithmetic
OF
LOGARITHMS.

CHAP. I.

Of the ORIGIN *and* NATURE *of* *LOGARITHMS.*

AS in Geometry the Magnitudes of Lines are often defined by Numbers; ſo, likewiſe, on the other Hand, it is ſometimes expedient to expound Numbers by Lines, *viz.* by aſſuming ſome Line which may repreſent Unity; the Double thereof, the Number 2; the Triple, 3; the one Half, the Fraction $\frac{1}{2}$; and ſo on. And thus the Geneſis and Properties of ſome certain Numbers are better conceived, and more clearly conſidered, than can be done by abſtract Numbers.

Hence, if any Line a * be drawn into itſelf, the Quantity a^2, produced thereby, is not to be taken as one of two Dimenſions, or as a Geometrical Square, whoſe Side is the Line a, but as a Line that is a third Proportional to ſome Line taken for Unity, and the Line a. So, likewiſe, if a^2 be multiplied by a, the Product a^3 will not be a Quantity of three Dimenſions, or a Geometrical Cube, but a Line that is the fourth Term in a Geometrical Progreſſion, whoſe firſt Term is 1, and ſecond a; for the Terms $1, a, a^2, a^3, a^4, a^5, a^6, a^7$, &c. are in the continual Ratio of 1 to a. And the Indices affixed to the Terms ſhew the Place or Diſtance that every Term is from Unity. For Example,

* *Fig.* 1.

ample, a^5 is in the fifth Place from Unity, a^6 in the ſixth, or ſix Times more diſtant from Unity than a, or a^1, which immediately follows Unity.

If, between the Terms 1 and a, there be put a mean Proportional, which is $\surd a$, the Index of this will be $\frac{1}{2}$; for its Diſtance from Unity will be one half of the Diſtance of a from Unity; and ſo $a^{\frac{1}{2}}$ may be written for $\surd a$. And if a mean Proportional be put between a and a^2, the Index thereof will be $1\frac{1}{2}$, or $\frac{3}{2}$; for its Diſtance will be ſeſquialteral of the Diſtance of a from Unity.

If there be two mean Proportionals put between 1 and a; the firſt of them is the Cube Root of a, whoſe Index muſt be $\frac{1}{3}$; for that Term is Diſtant from Unity only by a third Part of the Diſtance of a from Unity; and ſo the Cube Root may be expreſſed by $a^{\frac{1}{3}}$. Hence the Index of Unity is 0; for Unity is not diſtant from itſelf.

The ſame Series of Quantities, geometrically proportional, may be both Ways continued, as well deſcending towards the Left Hand, as aſcending towards the Right; for the Terms $\frac{1}{a^5}$, $\frac{1}{a^4}$, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, a^5, &c. are all in the ſame Geometrical Progreſſion. And ſince the Diſtance of a from Unity is towards the Right Hand, and poſitive or + 1, the Diſtance equal to that on the contrary Side, *viz.* the Diſtance of the Term $\frac{1}{a}$, will be negative, or — 1, which ſhall be the Index of the Term $\frac{1}{a}$, for which may be written a^{-1}. So likewiſe in the Terms a^{-2}, the Index —2 ſhews, that the Term ſtands in the ſecond Place from Unity towards the Left Hand, and the Expreſſions a^{-2} and $\frac{1}{a^2}$ are of the ſame Value. Alſo $a^{\;\;3}$ is the ſame as $\frac{1}{a^3}$. For theſe negative Indices ſhew, that the Terms belonging to them go from Unity the contrary Way to that by which the Terms, whoſe Indices are poſitive, do. Theſe Things premiſed,

If on the Line A N, both Ways indefinitely extended, be taken A C, C E, E G, G I, I L, on the Right Hand; and alſo AΓ, ΓΠ, *&c.* on the Left; all equal to one another; and if, at the Points Π, Γ, A, C, E, G, I, L, be erected to the Right Line A N, the Perpendiculars ΠΣ, ΓΔ, A B, C D, E F, G H, I K, L M, which let be continually proportional, and repreſent Numbers, whereof A B is Unity: The Lines A C, A E, A G, A I, A L,—AΓ,—AΠ, reſpectively expreſs the Diſtances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical Proportionals, according as it is diſtant from Unity. So ſince A G is triple of the Right Line A C, the Number G H ſhall be in the third Place from Unity, if C D be in the firſt: So likewiſe ſhall L M be in the fifth Place, ſince AL=5AC. If the Extremities of the Proportionals, Σ, Δ, B, D, F, H, K, M, be joined by Right Lines, the Figure ΣΠ L M will become a Polygon conſiſting of more or leſs Sides, according as there are more or leſs Terms in the Progreſſion.

If the Parts A C, C E, E G, G I, I L, be biſected in the Points *c*, *e*, *g*, *i*, *l*, and there be again raiſed the Perpendiculars *c d*, *e f*, *g h*, *i k*, *l m*, which are mean Proportionals between A B, C D; C D, E F; E F, G H; G H, I K; I K, L M; then there will ariſe a new Series of Proportionals, whoſe Terms, beginning from that which immediately follows Unity, are double of thoſe in the firſt Series, and the Differences of the Terms are become leſs, and approach nearer to a Ratio of Equality than before. Likewiſe in this new Series, the Right Lines A L, A C, expreſs the Diſtances of the Terms L M, C D, from Unity; *viz.* ſince A L is ten Times greater than A *c*, L M ſhall be the tenth Term of the Series from Unity: And becauſe A *e* is three Times greater than A *c*, *e f* will be the third Term of the Series, if *c d* be the firſt; and there ſhall be two mean Proportionals between A B and *e f*; and between A B and L M there will be nine mean Proportionals.

And if the Extremities of the ſaid Lines, *viz.* B, *d*, D, *f*, F, *h*, H, *&c.* be joined by Right Lines, there will be a new Polygon made, conſiſting of more, but ſhorter Sides than the laſt.

If, again, the Diſtances A *c*, *c* C, C *e*, *e* E, *&c.* be ſuppoſed to be biſected, and mean Proportionals between every two of the Terms be conceived to be put at thoſe middle Diſtances; then there will ariſe another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Difference of the Terms will be leſs; and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be leſſer, becauſe of the Diminution of the Diſtances of the Terms from each other.

Now, in this new Series, the Diſtances A L, A C, *&c.* will determine the Orders or Places of the Terms; *viz.* if A L be five Times greater than A C, and C D be the fourth Term of the Series from Unity, then L M will be the twentieth Term from Unity.

If in this manner mean Proportionals be continually placed between every two Terms, the Number of Terms at laſt will be made ſo great, as alſo the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon ſo leſſened, as to become leſs than any given Right Line; and conſequently the Polygon will be changed into a curve-lined Figure; for any curve-lined Figure may be conceived as a Polygon, whoſe Sides are infinitely ſmall, and infinite in Number.

A Curve deſcribed after this manner is called *Logarithmetical*; in which, if Numbers be repreſented by Right Lines ſtanding at Right Angles to the Axis A N, the Portion of the Axis intercepted between any Number and Unity ſhews the Place or Order, which that Number obtains in the Series of Geometrical Proportionals, diſtant from each other by equal Intervals. For Example; if A L be five Times greater than A C, and there are a thouſand Terms in continual Proportion, from Unity to L M; then will there be two hundred Terms of the ſame Series from Unity to C D, or C D ſhall be the two hundredth Term of the Series from Unity; and let the Number of Terms from A B to L M be ſuppoſed what it will, then the Number of Terms from A B to C D will be one fifth Part of that Number.

The Logarithmetical Curve may alſo be conceived to be deſcribed by two Motions, one of which is equable, and the other accelerated, or retarded, according to a given Ratio. For Example, if the Right Line AB moves uniformly along the Line AN, ſo that the End A thereof deſcribes equal Spaces in equal Times; and, in the mean Time, the ſaid Line AB ſo increaſes, that the Increments thereof, generated in equal Times, be proportional to the whole increaſing Line, that is, if AB, in going forward to cd, be increaſed by the Increment od, and in an equal Time when it is come to CD, the Increment thereof is Dp, and Dp to dc is as do is to AB; that is, if the Increments generated in equal Times are always proportional to the Wholes; or, if the Line AB, moving the contrary Way, diminiſhes in a conſtant Ratio, ſo that while it goes thro' the equal Spaces, the Decrements $AB - \Gamma\Delta$, $\Gamma\Delta - \Pi\Sigma$, are proportional to AB, $\Gamma\Delta$; then the End of the Line, increaſing or decreaſing in the ſaid manner, deſcribes the Logarithmetical Curve. For ſince $AB : do :: dc : Dp :: DC : fq$; it ſhall be (by Compoſition of Ratio), as $AB : dc :: dc : DC :: DC : fe$, and ſo on.

By theſe two Motions, *viz.* the one equable, and the other proportionally accelerated or retarded, the Lord *Neper* laid down the Origin of Logarithms, and called the Logarithm of the Sine of any Arc, *That Number which neareſt defines a Line that equally increaſes, while, in the mean Time, the Line expreſſing the whole Sine proportionally decreaſes to that Sine.*

It is manifeſt, from this Deſcription of the Logarithmetic Curve, that all Numbers at equal Diſtances are continually proportional. It is alſo plain, that if there be four Numbers AB, CD, IK, LM, ſuch, that the Diſtance between the firſt and ſecond be equal to the Diſtance between the third and the fourth: Let the Diſtance from the ſecond to the third be what it will, theſe Numbers will be proportional. For, becauſe the Diſtances AC, IL, are equal, AB ſhall be to the Increment Ds, as IK is to the Increment MT. Wherefore (by Compoſition) $AB : DC : IK : ML$. And contrariwiſe, if four Numbers be proportional, the Diſtance between the firſt and the ſecond ſhall be equal to the Diſtance between the third and the fourth.

The Diſtance between any two Numbers is called the Logarithm of the Ratio of thoſe Numbers, and indeed doth not meaſure the Ratio itſelf, but the Number of Terms in a given Series of Geometrical Proportionals proceeding from one Number to another; and defines the Number of equal Ratios by the Compoſition whereof the Ratios of Numbers are known.

If the Diſtance between any two Numbers be double to the Diſtance between two other Numbers, then the Ratio of the two former Numbers ſhall be the Duplicate of that Ratio of the two latter. For let the Diſtance I L between the Numbers I K, L M, be double to the Diſtance A *c*, between the Numbers A B, *c d*; and ſince I L is biſected in *l*, we have A *c*=I *l*,= *l* L; and the Ratio of I K to *l m* is equal to the Ratio of A B to *c d*; and ſo the Ratio of I K to L M, the Duplicate of the Ratio of I K to *l m* (*by Def.* 10. *El.* 5.), ſhall be the Duplicate of the Ratio of A B to *c d*.

In like manner, if the Diſtance E L be triple of the Diſtance A C, then will the Ratio of E F to L M be triplicate of the Ratio of A B to C D: For, becauſe the Diſtance is triple, there ſhall be three Times more Proportionals from E F to L M, than there are Terms of the ſame Ratio from A B to C D; and the Ratio of E F to L M, as alſo of A B to C D, is compounded of the equal intermediate Ratios (*by Def.* 5. *El.* 6.) And ſo the Ratio of E F to L M, compounded of three Times a greater Number of Ratios, ſhall be triplicate of the Ratio of AB to C D. So, likewiſe, if the Diſtance G L be quadruple of the Diſtance A *c*, then ſhall the Ratio of G H to L M be quadruplicate of the Ratio of A B to *c d*.

The Logarithm of any Number is the Logarithm of the Ratio of Unity to that Number; or it is the Diſtance between Unity and that Number. And ſo Logarithms expreſs the Power, Place, or Order, which every Number, in a Series of Geometrical Progreſſionals, obtains from Unity. For Example, if there be 10000000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10000000th Place from Unity; then it will be found by Computation, that in the ſame Series from Unity, to 2, there are 3010300 proportional Terms; that is, the Number 2 will ſtand in the 3010300th Place. In like manner, from Unity to 3, there will be found 4771213 proportional Terms, which Number defines the Place of the Number 3. The Numbers 10000000, 3010300, 4771213, ſhall be the Logarithms of the Numbers 10, 2, and 3.

If the firſt Term of the Series from Unity be called y, the ſecond Term will be y^2, the third y^3, *&c.* And ſince the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000} = 10$; alſo $y^{3010300} = 2$; alſo $y^{4771213} = 3$; and ſo on.

Wherefore all Numbers ſhall be ſome Powers of that Number which is the firſt from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Diſtances of Numbers from Unity, as has been ſhewn, the Logarithm of Unity ſhall be 0; for Unity is not diſtant from itſelf: But the Logarithms of Fractions are negative, or deſcending below nothing; for they go on the contrary Way. And ſo if Numbers, increaſing proportionally from Unity, have poſitive Logarithms, or ſuch as are affected with the Sign +, then Fractions or Numbers, in like manner decreaſing, will have negative Logarithms, or ſuch as are affected with the Sign —; which is true when Logarithms are conſidered as the Diſtances of Numbers from Unity.

But if Logarithms take their Beginning, not from an integral Unit, but from an Unit that is in ſome Place of decimal Fractions; for Example, from the Fraction $\frac{1}{10000000000}$; then all Fractions greater than this, will have poſitive Logarithms; and thoſe that are leſs, will have negative Logarithms. But more ſhall be ſaid of this hereafter.

Since in the Numbers continually proportional, C D, E F, G H, I K, *&c.* the Diſtances C E, E G, G I, *&c.* are equal, the Logarithms A C, A E, A G, A I, *&c.* of thoſe Numbers ſhall be equidifferent, or the Differences of them ſhall be equal: And ſo the Logarithms of proportional Numbers are all in an Arithmetical Progreſſion; and from hence proceeds that common Definition of Logarithms, *viz. that Loga-*

rithms are Numbers, which, being adjoined to Proportions, have equal Differences.

In the firſt Kind of Logarithms that *Neper* publiſhed, the firſt Term of the continual Proportionals was placed only ſo far diſtant from Unity, as that Term exceeded Unity. For Example, if vn be the firſt Term of the Series from Unity AB, the Logarithm thereof, or the Diſtance An, or By, was, according to him, equal to vy, or the Increment of the Number above Unity. As ſuppoſe vn be 1,0000001, he placed 0,0000001 for its Logarithm An; and from hence, by Computation, the Number 10 ſhall be the 23025850th Term of the Series; which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expreſſes its Diſtance from Unity in ſuch Parts whereof vy or An is one.

But this Poſition is entirely at Pleaſure; for the Diſtance of the firſt Term may have any given Ratio to the Exceſs thereof above Unity; and according to that various Ratio (which may be ſuppoſed at Pleaſure,) that is, between vy and By, the Increment of the firſt Term above Unity, and the Diſtance of the ſame from Unity, there will be produced different Forms of Logarithms.

This firſt Kind of Logarithms was afterwards changed by *Neper*, into another more convenient one, wherein he put the Number 10 not as the 23025850 Term of the Series but the 10000000th; and in this Form of Logarithms, the firſt Increment vy ſhall be to the Diſtance By, or An, as Unity, or AB, is to the Decimal Fraction 0,4342994, which therefore expreſſes the Length of the Subtangent AT, *Fig.* 4.

After *Neper*'s Death, the excellent Mr. *Henry Briggs*, by great Pains, made and publiſhed Tables of Logarithms according to this Form. Now ſince in theſe Tables, the Logarithm of 10, or the Diſtance thereof from Unity, is 1,0000000, and 1, 10, 100, 1000, 10000, *&c.* are continual Proportionals, they ſhall be equidiſtant. Wherefore the Logarithm of the Number 100 ſhall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10000 ſhall be 4,0000000; and ſo on.

Hence the Logarithms of all Numbers between 1 and 10 muſt begin with 0, or 0 muſt ſtand in the firſt Place to the Left-hand; for they are leſſer than the Logarithm of the Number 10, whoſe Beginning is Unity; and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and leſs than 2,0000000. Alſo the Logarithms between 100 and 1000 begin with 2; for they are greater than the Logarithm of 100 which begins with 2, and leſs than the Logarithm of 1000 that begins with 3. In the ſame manner it is demonſtrated, that the firſt Figure to the Left-hand of the Logarithms between 1000 and 10000 muſt be 3; and the firſt Figure to the Left-hand of the Logarithms between 10000 and 100000 will be 4; and ſo on.

The firſt Figure of every Logarithm to the Left-hand is called the Characteriſtic, or Index, becauſe it ſhews the higheſt or moſt remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the higheſt or moſt remote Place from Unity of the correſpondent Number, to the Left-hand, will be the Place of Tens. If the Index be 2, the moſt remote Figure of the correſpondent Number ſhall be in the ſecond Place from Unity, that is, it ſhall be in the Place of Hundreds; and if the Index of a Logarithm be 3, the laſt Figure of the Number anſwering to it, ſhall be in the Place of Thouſands. The Logarithms of all Numbers that are in decuple or ſubdecuple Progreſſion, only differ in their Characteriſtics, or Indices, they being written in all other Places with the ſame Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17000, are the ſame, unleſs in their Indices; for ſince 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17000; therefore the Diſtances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17000, ſhall be all equal. And ſo, ſince the Diſtance between 1 and 17, or the Logarithm of the Number 17 is 1.2304489, the Logarithm of the Number 170 will be=2.2304489, and the Logarithm of the Number 1700 ſhall be 3.2304489, becauſe the Logarithm of the Number 100=2.0000000. In like manner, ſince the Logarithm of the Number 1000=3.0000000, the Logarithm of the Number 17000 ſhall be 4.2304489.

So alſo the Numbers, 6748. 674,8. 67,48. 6,748. 0,6748. 0,06748, are continual Proportions in the Ratio of 10 to 1; and ſo their Diſtances from each other ſhall be equal to the Diſtance or Logarithm of the Number 10, or equal to 1,0000000. And ſo, ſince the Logarithm of the Number 6748 is 3,8291751, the Logarithms of the other Numbers ſhall be as in the Margin; where you may obſerve, that the Indices of the laſt two Logarithms are only negative, and the other Figures poſitive; and ſo, when thoſe other Figures are to be added, the Indices muſt be ſubtracted, and contrariwiſe.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	—1,8291751
0,06748	—2,8291751

CHAP. II.

Of the Arithmetic of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BEcauſe, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product; the Diſtance between Unity and the Multiplier ſhall be equal to the Diſtance between the Multiplicand and the Product. If therefore the Number G H be to be multiplied by the Number E F, the Diſtance between G H and the Product muſt be equal to the Diſtance A E, or to the Logarithm of the Multiplier; and ſo, if G L be taken equal to A E, the Number L M ſhall be the Product; that is, if the Logarithm of the Multiplicand A G be added to the Logarithm of the Multiplier A E, the Sum ſhall be the Logarithm of the Product.

In Diviſion, the Diviſor is to Unity, as the Dividend is to the Quotient; and ſo the Diſtance between the Diviſor and Unity ſhall be equal to the Diſtance between the Dividend and the Quotient. So if L M be to be divided by E E, the Diſtance E A ſhall be equal to the Diſtance between L M and the Quotient;

and ſo, if L G be taken equal to E A, the Quotient will be at G; that is, if from A L, the Logarithm of the Dividend, be taken G L, or A E, the Logarithm of the Diviſor, there will remain A G, the Logarithm of the Quotient.

And from hence it appears, that whatſoever Operations in common Arithmetic are performed by multiplying or dividing of great Numbers, may be done much eaſier, and more expeditiouſly, by the Addition or Subtraction of Logarithms.

For Example, Let the Number 7589 be to be multiplied by 6757. Now, if the Logarithms of thoſe Numbers be added together, as in the Margin, their Sum will be the Logarithm of the Product, whoſe Index 7 ſhews, that there are ſeven Places of Figures, beſides Unity, in the Product; and in ſeeking this Logarithm in Tables, or the neareſt equal to it, I find that the Number anſwering thereto, which is leſſer than the Product, is 51278000; and the Number greater than the Product is 51279000; and if the adjoined Differences, and proportional Parts, be taken, the Numbers that muſt be added to the Place of Hundreds and Tens in the Product are 87; and that which muſt be added in the Place of Unity, will neceſſarily be 3, ſince ſeven Times 9=63; and ſo the true Product ſhall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers to be added in the Place of Tens or Hundreds could not be had from thoſe Tables of Logarithms which conſiſt of but 7 Places of Figures, beſides the Characteriſtic; and ſo, in this Caſe, the *Valacquian* or *Briggian* Tables ſhould be uſed; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

Log. 3.8801846
Log. 3.8297539
Log. 7.7099385

If the Number 78596 be to be divided by 276, by ſubtracting the Logarithm of the Diviſor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 anſwers; which therefore ſhall be the Quotient.

Log. 4.8954004
Log. 2.4440448
Log. 2.4513556

Becauſe Unity, any aſſumed Number, the Square thereof, the Cube the Biquadrate, &c. are all continual Proportionals, their Diſtances from each other ſhall be equal to one another. And ſo it is manifeſt, that the Diſtance of the Square from Unity is double of the Diſtance of its Root from the ſame: Alſo the Diſtance of the Cube is triple of the Diſtance of its Root; and the Diſtance of the Biquadrate is quadruple of the Diſtance of its Root from Unity, &c. And ſo, if the Logarithm of any Number be doubled, we ſhall have the Logarithm of its Square; if it be tripled, we ſhall have the Logarithm of its Cube; and if it be quadrupled, the Logarithm of its Biquadrate. And contrariwiſe, if the Logarithm of any Number be biſected, we ſhall have the Logarithm of the Square Root thereof: Moreover, a third Part of the ſaid Logarithm will be the Logarithm of the Cube Root of the Number; and a fourth Part, the Logarithm of the Biquadrate Root of that Number.

Hence, the Extractions of all Roots are eaſily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square Root of 5, the Half of 0,6989700 muſt be taken, and then that Half 0,3494850 will be the Logarithm of the Square Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2,236068 nearly anſwers.

CHAP. III.

Of the Arithmetic of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is neceſſary, for avoiding the Trouble of adding one Part of a Logarithm, and ſubtracting the other, that Logarithms do not begin from an integral Unit, but from ſome Unit that is in the Tenth or Hundredth Place of Decimal Fractions: For Example, let P O be $\frac{1}{10000000000}$, and from this let the Logarithm begin. Now this Fraction is ten Times more diſtant from Unity to the Left-hand, than the Number 10 is diſtant therefrom to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to P O. And ſo, if [illegible]ry, the Logarithm thereof, according to this Suppoſition, will not be 0, but O A=10.0000000. Now the Diſtance of Ten from Unity is 1.0000000, whence the Diſtance of the Number 10 from P O will be 11.0000000. Alſo the Diſtance of the Number 100 from P O, or its Logarithm, beginning from P O, ſhall be 12.0000000; and the Logarithm of 1000, or the Diſtance from P O, will be 13.0000000. And thus, the Indices of all Logarithms are augmented by the Number 10; and thoſe Fractions whoſe Indices are —1, or—2, or—3, *&c.* are now made 9, 8, or 7, *&c.*

But if Logarithms begin from the Place of a Fraction, whoſe Numerator is Unity, and Denominator Unity with 100 Cyphers added to it (which they muſt do when Fractions occur that are leſs than P O), then that Fraction will be 100 Times more diſtant from Unity, than 10 is diſtant from it; and ſo the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and ſo on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than P O (whereat they begin) will be poſitive. And ſince the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c.* are in a continual Geometrical Progreſſion, they will be equally diſtant from each other; and accordingly their Logarithms will be equidifferent: And ſo, when the Logarithm of 10 is 11.0000000, and the Logarithm of Unity is 10.0000000; then the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000; and, in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Alſo, for the ſame Reaſon, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of the Logarithm of the Fraction $\frac{1}{1000}$ ſhall be 97, *&c.* And theſe Indices ſhew in what Place from Unity the firſt Figure of the Fraction, not being a Cypher, muſt be put. For Example, if the Index be 4, the Diſtance thereof from the Index of Unity (which is 10), *viz.* 6, ſhews that the firſt ſignificative Figure of the Decimal is in the ſixth Place from Unity; and therefore five Cyphers are to be prefixed

thereto

thereto towards the Left-hand. So, alſo, if the Index of Unity be 100, and the Index of the Fraction be 80, the firſt Figure thereof ſhall be in the 20th Place from Unity, and 19 Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction G H by the Fraction D C. Becauſe Unity is to the Multiplier as the Multiplicand is to the Product; the Diſtance between Unity and the Multiplier ſhall be equal to the Diſtance between the Multiplicand and the Product. Therefore, if there be taken G I = A C, the Product I K ſhall be at I. And, accordingly, if from O G, the Logarithm of the Multiplicand, there be taken G I or A C, there will remain O I, the Logarithm of the Product. But A C = O A — O C, which taken from O G, there will remain O G + O C — O A = O I; that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity (which is always expreſſed by 10 or 100 with Cyphers), the Logarithm of the Product will be had. For Example, let the Decimal Fraction 0,00734 be to be multiplied by the Fraction 0,000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margin; which being added together, and the Logarithm of Unity being taken away from the Sum, the Remainder is the Logarithm of the Product, whoſe Index 94 ſhews, that the firſt Figure of the Product is in the ſixth Place from Unity; and ſo there muſt be five Cyphers prefixed, and then the Product will be 0,00000642984.

97,8656961
96.9425041
94,8082002

In Diviſion, the Diviſor is to Unity, as the Dividend is to the Quotient; and ſo the Diſtance between the Diviſor and Unity ſhall be equal to the Diſtance between the Dividend and Quotient. And ſo, if the Fraction I K be to be divided by D C, you muſt take I G = C A, and the Place of the Quotient ſhall be G. But C A = O A — O C, which being added to O I, we have O A + O I — O C = O G; that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Diviſor, there will remain the Logarithm of the Quotient; ſo if the Number C D be to be divided by IK, you muſt take the Diſtance $CS = IA$, and then ST will be the Quotient, whoſe Logarithm is $OA + OC - OI$. Let $CD = 0.347$, $IK = 0.00478$. Then add the Logarithm of Unity to the Logarithm of CD; that is, put 1 or 10 before the Index thereof, and from that ſubtract the Logarithm of the Diviſor, and the Remainder will be the Logarithm of the Quotient, whoſe Index 11 ſhews, that the Quotient is between the Numbers 10 and 100; and I ſeek the Number anſwering the Logarithm, which I find to be 72,594. If the Logarithm of a Vulgar Fraction, for Example $\frac{7}{8}$, be required, the Logarithm of Unity muſt be added to the Logarithm of the Numerator 7; or, which is all one, you muſt put 10 or 100 before the Index thereof, and ſubduct from it the Logarithm of the Denominator 8; and there will remain the Logarithm of the Vulgar Fraction $\frac{7}{8}$, or the Decimal .875.

$$\begin{array}{r} 19.5403295 \\ 7.6794279 \\ \hline 11.8609016 \end{array}$$

$$\begin{array}{r} 10.8450980 \\ 0.9030900 \\ \hline 9.9420080 \end{array}$$

If the Powers of any Fraction DC be required, you muſt aſſume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually proportional from Unity. Beſides, $AE = 2AC = 2AO - 2OC$; whence $OE = OA - AE = 2OC - OA$; that is, the Logarithm of the Square is the Double of the Logarithm of the Root, leſs the Logarithm of Unity. In like manner, ſince $AG = 3AC = 3OA - 3OC$, we ſhall have $OG = OA - AG = 3OC - 2OA =$ the Logarithm of the Cube, $=$ triple the Logarithm of the Root,—the Double of the Logarithm of Unity. For the ſame Reaſon, becauſe $AI = 4AC = 4OA - 4OC$, we have $OI = 4OC - 3OA$, which is the Logarithm of the Biquadrate. And, univerſally, if the Power of a Fraction be n, and the Logarithm L, then ſhall the Logarithm of the Power $n = nL - nOA + OA$; that is, if the Logarithm of a Fraction be multiplied by n, and from the Product be taken the Logarithm of Unity, multiplied by $n - 1$, the Logarithm of the Power n of that Fraction will be had.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{20}$=,05, the Logarithm of this Fraction 8.6989700, which, being multiplied by 6, gives the Number 52.1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, *viz.* 2 1938200, to which the Number ,000000015625 anſwers. For the Index 2 ſhews, that 7 Cyphers muſt be put before the firſt Figure.

If the 8th Power of the Fraction ,05 be required, by multiplying the Logarithm by 8, there will be produced 69.5917600; and ſince 70, which is ſeven Times the Index of the Logarithm of Unity, cannot be taken from 69, unleſs we run into negative Numbers, the Index of the Logarithm of Unity muſt be ſuppoſed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm drawn into 8, gives 789,5917600; and if 700, which is 7 Times the Index of the Logarithm of Unity, be taken from 789, there will remain 89.5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whoſe correſpondent Number is ,000000000039062. For ſince the Index is 89, and the Difference thereof from 100 is 11; the firſt ſignificative Figure of the Fraction ſhall be in the 11th Place from Unity; and ſo there muſt be 10 Cyphers placed before it.

If the Roots of the Powers of Fractions be deſired, for Example, the Square Root of the Fraction E F; becauſe the Root is a mean Proportional between the Fraction and Unity, you muſt biſect A E in C, and then C D will be the Square Root of the Fraction E F. But $AC = \frac{1}{2}\ AE = \frac{OA-OE}{2}$; and ſo the Logarithm of the Root $= OA-AC = \frac{OA+OE}{2}$. And if the Cube Root of the Fraction G H be ſought, this ſhall be the firſt of two mean Proportionals between Unity and G H; and ſo, if A G be divided into three equal Parts, the firſt of which is A C, then C D ſhall be the Root ſought: And becauſe $AC = \frac{1}{3} AG = \frac{OA-OG}{3}$, if this be taken from OA, there will remain $\frac{2OA+OG}{3} = OC =$ Logarithm of the Cube Root of the Fraction GH. So, likewiſe, the Biquadrate Root of the Fraction IK will be had, by dividing AI into four equal Parts; for the Root is the firſt of three mean Proportionals between Unity and the Fraction; and, conſequently, if $AC = \frac{1}{4}AI$, then will CD be the Biquadrate Root of the Fraction IK. But $AC = \frac{1}{4}AI = \frac{OA-OI}{4}$; and ſo $OC = OA-AC = \frac{3OA+OI}{4}$.

And univerſally, if the Root of any Power n of the Fraction LM be required, the Logarithm of the Root thereof will be $\frac{nOA-OA+OL}{n}$; that is, if the Number $n-1$ be prefixed to the Index of the Logarithm, and the Logarithm thus augmented be divided by n, the Quotient will give the Logarithm of the Root ſought. So if the Cube Root of the Fraction $\frac{1}{2}$ or .5 be ſought, you muſt place $2=n-1$ (ſince the Cube Root is required) before the Logarithm thereof, and there will be had 29.6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$; and the Number ,7937, anſwering to this Logarithm, is the Root ſought.

CHAP. IV.

Of the Rule of Proportion by Logarithms.

THE Rule of Proportion ſhews how, by having three Numbers given, a fourth Proportional to them may be found; *viz.* if the ſecond and third Terms be multiplied by one another; and the Product divided by the firſt Term, then will the Quotient be the fourth proportional Term ſought. But this fourth Term is much eaſier found by Logarithms; for if the Logarithm of the firſt Term be taken from the Sum of the Logarithms of the ſecond and third Term, the Number remaining will be the Logarithm of the fourth, ſought.

Or this may be done ſomething eaſier yet, if inſtead of the Logarithm of the firſt Term be taken its Complement Arithmetical, or the Difference of that Logarithm, and the Number 10.0000000, which is done by ſetting down the Difference between each Figure of the Logarithm, and the Figure 9; for then, if that Arithmetical Complement be added to the Sum of the other two Logarithms; and if Unity, which is the firſt Figure to the Left-hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term ſought; and ſo, by this Way, the Logarithm of the fourth Term is found by only one Addition of three Numbers. The Reaſon of this will be manifeſt from hence: Let there be three Numbers A, B, C, the firſt of which is to be taken from the Sum of the ſecond and third. Now this may not only be done by the common Way, but, likewiſe, if there be any other Number E taken, and from this there be taken A, there will remain E—A; and if the Numbers B, C, and E—A, be all added together, and from their Sum be taken E, there will remain B+C—A. So, if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100 which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

85
23
108

Here follow ſome Trigonometrical Examples of the Rule of Proportion ſolved by Logarithms.

Let A B C be a Right-lined Triangle, wherein are given the Angle A 36 Degrees 46′, the Angle B 98 Degrees 32′, and the Side B C 3478; the Side A C is required. Say (*by Caſe* 1. *of Plane Trig.*), as the Sine of the Angle A is to the Sine of the Angle B, ſo is B C to A C. And becauſe the Logarithm Sine of the Angle A is the firſt Term of the Analogy, I ſubſtitute its Complement Arithmetical for the ſame, and add the Logarithm of BC, the Logarithm of S, B, and the ſaid Complement, all three together, and reject Unity

Arith. Comp. S, A.	0.2228938
Log. Sin. B.	9.9951656
Log. BC.	3.5413296
Log. AC.	13.7593890

Unity, which is in the firſt Place to the Left-hand; and then the Logarithm of the Side A C will be given, and the Number anſwering thereto is 5746,306, equal to the Side ſought A C.

Let there be a Spherical Triangle A B C, in which are given all the Sides, *viz.* B C = 30 Degrees, A B = 24 Degrees 3', and A C = 42° Degrees 8'; the Angle B is required. Let B A be produced to M, ſo that B M = B C; then will A M, the Difference of the Sides B C, B A, be equal to 5 Degrees 57'. Now (*by Caſe* II. in oblique-angled Spherical Triangles) ſay, As the Rectangle under the Sines of the Legs is to the Square of Radius, ſo is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$, to the Square of the Sine of one half the Angle B.

But $\frac{AC+AM}{2}$ = 24 Degrees 2', and $\frac{AC-AM}{2}$ = 18 Degrees 6'; and becauſe the firſt Term of the Analogy is the Rectangle under the Sines of A B, B C, and the ſecond Term is the Square of Radius, the Sum of the Logarithm Sines of A B, B C, muſt be taken from double the Logarithm of Radius, and what remains muſt be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$, which is the ſame as if the Logarithm Sines of each of the Arcs A B, B C, were ſubtracted from the Log. of Radius; or if the Complements Arithmetical of theſe Sines be taken, and thoſe Complements and the ſaid Sines be all added together, then ſhall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And ſo the Half thereof, *viz.* the Logarithm 9.8965274, is the Logarithm Sine of half the Angle B = 51 Degrees 59' 56", and the Double of this Angle ſhall be 103 Degrees 59' 52" = B, which was ſought.

Log. S, BC Comp. Arith.	0.3010299
Log. S, AB Comp. Arith.	0.3898364
Log. S, $\frac{AC+AM}{2}$	9.6098803
Log. S, $\frac{AC-AM}{2}$	9.4923083
2 Log. S $\frac{1}{2}$ Angle B	19.7930549

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms, any Term in a Series of Proportionals, either increaſing or decreaſing. Fig. 3.

IF any where in the Axis of the Logarithmetical Curve, there be taken any Number of equal Parts SV, VY, YQ, *&c.* and at the Points S, V, Y, Q, *&c.* be raiſed the Perpendiculars ST, VX, YZ, Qπ, *&c.* then, from the Nature of the Curve, ſhall all theſe Perpendiculars be continually proportional; and therefore, alſo, the continual Increments X*x*, Z*z*, πϖ, ſhall be proportional to their Wholes. For ſince ST : VX :: VX : YZ :: YZ : Qπ, it ſhall be (by Diviſion of Proportion) ST : X*x* :: VX : Z*z* :: YZ : πϖ; and (by Compoſition of Proportion) VX : X*x* :: YZ : Z*z* :: Qπ : πϖ. Hence if X*x* be any Part of any Right Line ST, then will Z*z* be the ſame Part of the Right Line VX, and alſo πϖ the ſame Part of the Right Line YZ. For Example, if X*x* be the $\frac{1}{20}$ Part of ST, then will $Zz=\frac{1}{20}VX$, and $\pi\varpi=\frac{1}{20}YZ$; or, which comes to the ſame, we ſhall have $VX=ST+\frac{1}{20}ST$, $YZ=VX+\frac{1}{20}VX$, alſo, $Q\pi=YZ+\frac{1}{20}YZ$.

Now make, as ST is to VX, ſo is Unity AB to NR; then ſhall AN = SV; and ſo each of the Right Lines SV, VY, YQ, *&c.* ſhall be equal to the Logarithm of RN; and AV, the Logarithm of the Term VX, ſhall be equal to AS + AN = Logarithm of ST + Logarithm of NR. Alſo AY, the Logarithm of the Term YZ, ſhall be equal to AS + 2AN = Logarithm ST + 2 Logarithm NR; and AQ, the Logarithm of the Term Qπ, ſhall be equal to AS + 3AN = Logarithm ST + 3 Logarithm NR. And univerſally, if the Logarithm of the Number NR be multiplied by a Number, expreſſing the Diſtance of any Term from the firſt, and the Product be added to the Logarithm of the firſt Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreaſing, that is, if the Terms diminiſh in a continual Ratio, and Qπ be the firſt Term; then the Logarithm of any other will be had, by multiplying the Logarithm of the Number N R, by a Number that expreſſes the Diſtance of its Term from the firſt, and ſubtracting the Product from the Logarithm of the firſt. And if the ſaid Product be greater than the Logarithm of the firſt Term, then the Logarithms muſt begin from a Unit in ſome Place of Decimal Fractions, as from O P, and then the Logarithm of the Number Qπ will be O Q.

Now, let L M repreſent any Money, or Sum of Money, put out to Intereſt, ſo that the Intereſt thereof be accounted but at the End of every Year, and let K *k* be the Gain or Intereſt thereof at the End of the firſt Year; then will I K be the Sum of the Intereſt and Principal. And again I K, becoming the Principal at the End of the firſt Year, H *b*, which is proportional to I K, or in a conſtant Ratio, will be the Gain at the End of the ſecond Year; and ſo H G, at the End of the ſecond Year, will become the Principal; and at the End of the third Year F *f*, proportional to G H, will be the Gain. Now let us ſuppoſe the Principal to be augmented every Year $\frac{1}{20}$ Part thereof, ſo that $IK = LM + \frac{1}{20} LM$, $GH = IK + \frac{1}{20} IK$, $EF = GH + \frac{1}{20} GH$, and ſo on. And accordingly, the Terms LM, I K, G H, E F, *&c.* are continual Proportionals, and it is required to find the Amount of the Money at the End of any Number of Years.

Let L M be a Farthing. Becauſe L M is to I K as 1 to $1 + \frac{1}{20}$, or as 1 to 1.05, as A B is to N R, then will $NR = 1.05$, whoſe Logarithm A N, is 0.0211893, or, more accurately, 0.021892991, it is required to find the Amount of a Farthing, put out at Compound Intereſt, at the End of 600 Years. Multiply A N by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$, *viz.* 97.0177288 (for a Farthing is $\frac{1}{960}$ Part of a Pound), and the Sum 109.7313082 ſhall be the Logarithm of the Number ſought; and ſince the Index 109 exceeds the Index of Unity by 9, there ſhall be nine Places of Figures above Unity in the correſpondent Number;

ber; and that Number, being ſought in the Tables, will be found greater than 5386500000, and leſs than 5386600000. And therefore a Farthing put out, at Intereſt upon Intereſt, at 5 *per Cent. per Annum*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up, by all the Gold and Silver that has been dug out of the Bowels of the Earth, from the Beginning of the World to this Time.

Let Qπ expound any Sum of Money due to ſome Perſon at the End of a full Year. Now it is certain, that if the Debtor ſhould pay down, at preſent, the whole Sum of Money, he would loſe the yearly Uſury or Intereſt that his Money would gain him; and ſo a leſſer Sum, being put out to Intereſt, will, at the End of one Year, together with the Intereſt thereof, be equal to the Sum of Money Qπ. Now this preſent Sum of Money, which, together with the Intereſt thereof, is equal to the Sum of Money Qπ, is called the preſent Worth of the Money Qπ. Let AN be the Logarithm of the Ratio which the Principal has to the Sum of the Principal and Intereſt, that is, if the Principal be twenty Times the yearly Intereſt, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QY equal to AN; then will AY be the Logarithm of the preſent Worth of the Money Qπ. For it is manifeſt, that the Money YZ put out to Intereſt, will, at the End of one Year, amount to the Money Qπ; and ſo, to have the Logarithm of the preſent Worth thereof, or YZ, the Logarithm AN muſt be taken from the Logarithm AQ, and there will remain the Logarithm AY of the preſent Worth, or YZ. But if the Sum Qπ be not due till the End of two Years, then the Logarithm 2AN muſt be ſubtracted from the Logarithm AQ, and there will remain AV, the Logarithm of the preſent Worth, or of the Sum that muſt be paid at preſent for the Money Qπ due at the End of two Years. For it is manifeſt, that the Money VX being put out to Intereſt, will, at the End of two Years amount to the Sum of Money Qπ. By the ſame Reaſon, if the Sum Qπ be not due until the End of three Years, the Logarithm 3AN muſt be ſubtracted from the Logarithm of Qπ, and the Remainder AS ſhall be the Logarithm of the Number ST, or

ST

S T ſhall be the preſent Worth of the Sum Q π due at the three Years End. And univerſally, if the Logarithm A N be multiplied by the Number of Years, at the End of which the Sum Q π is due, and the Number produced be taken from the Logarithm A Q, then will the Logarithm of the preſent Worth of the Sum Q π be had. And from hence it is manifeſt, if 5386500000 Pounds be due to ſome Society at the End of 600 Years, then would the preſent Worth of that vaſt Sum of Money be ſcarcely a Farthing.

If the proportional Right Lines H G, E F, A B, C D, *Fig.* 4. are Ordinates to the Axis of the Logarithmetical Curve, and if their Ends F H, D B, be joined by Right Lines, which, produced, meet the Axis in the Points P and K, then the Right Lines G P, A K, will be always equal. For ſince G H : E F : : A B : C D; it will be, as G H : F *s* : : A B : D R. But becauſe of the equiangular Triangles P G H, H *s* F, as alſo K A B, B R D, we have P G : H *s* : : (G H : F *s* : : A B : D R : :) K A : B R. And ſince the Conſequents H *s*, B R, are equal, the Antecedents P G, K A, ſhall be alſo equal. W. W. D.

If the Right Lines C D, E F, equally accede to A B, G H, ſo that the Point D at laſt may coincide with B, and the Point F with H, then the Right Lines D B K, F H P, which did cut the Curve before, will be changed into the Tangents B T, H V. And the Right Lines A T, G V, will be always equal to each other; that is, the Portion of the Axis A T, or G V, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every where be of a conſtant and given Length. And this is one of the chief Properties of the Logarithmetical Curve; for the different Species or Forms of thoſe Curves are determined by the Subtangents.

The Logarithms, or the Diſtances from Unity of the ſame Number, in two Logarithmetick Curves of different Species, will be proportional to the Subtangents of their Curves. For let H B D, S N Y, *Fig.* 4, 5. be Curves, whoſe Subtangents are A T, M X, and let A B = M N = Unity; alſo, D C = Q Y; then ſhall A C, the Logarithm of the Number C D, in the Logarithmetick Curve H D, be to M Q, the Logarithm of the Number Q Y (or of the ſaid C D), in the Curve S Y; as the Subtangent A T is to the Subtangent M X.

MX. For let there be ſuppoſed an infinite Number of mean Proportional Terms between AB, CD, or MN, QY, in the Ratio of AB to ab, or MN to mn; and ſince $AB = MN$, then will $ab = mn$, as alſo $bc = no$. And becauſe the Number of proportional Terms in each Figure are equal, they do divide the Lines AC, MQ, into equal Numbers of Parts, the firſt of which are Aa, Mm, and ſo the ſaid Parts ſhall be proportional to their Wholes; that is, it will be as $Aa : Mm :: AC : MQ$. And becauſe the Triangles TAB, Bcb, are ſimilar (for the Part of the Curve Bb nearly coincides with the Portion of the Tangent), as alſo the Triangles XMN, Non, we have Aa, or $Bc : bc :: TA : AB$.

Alſo, as no, or $bc : No :: MN$, or $AB : MX$.

Where (by Equality of Proportion) it will be, $Bc : No :: TA : MX :: Aa : Mm :: AC : MQ$; *which was to be demonſtrated.* If AT be called a, ſince $AB : AT :: bc : Bc$, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a ſmall Matter exceeding it, be given, then will the Subtangent of the Logarithmetick Curve be had. For the Exceſs bc is to the Logarithm Bc, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference ſhall be to the Difference of the Logarithms, as one of the Numbers is to the Subtangent. For Example, if the Increment bc be ,00000 00000 00001 02255 31945 60259, and Bc or Aa the Logarithm of the Number ab be ,00000 00000 00000 44408 92098 50062. Now if a fourth Proportional be found to the ſaid two Numbers and Unity, *viz.* 434294481903251, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve expreſſing *Briggs*'s Logarithms.

If a Sum of Money be put out to Intereſt on this Condition, that a proportional Part of the yearly Rate of Intereſt thereof be accounted every Moment of Time, *viz.* ſo that at the End of the firſt Moment of Time, or indefinitely ſmall Particle of a Year, the Intereſt gotten thereby be proportional to that Time; which being added to the Principal, again begets Intereſt at the End of the ſecond Moment of Time, and then the Principal and this Intereſt become a Principal, and ſo on; it is required to find the Amount of that Sum at the Year's End. Let a be nearly the Intereſt of Unity, or of one Pound. Then, if one whole Year, or 1, gives the Intereſt a, the indefinitely ſmall Particle of a Year Mm will give the Intereſt $Mm \times a$, proportional to Mm; and, accordingly, if Unity be expounded by MN, the firſt Increment thereof ſhall be $no = Mm \times a$. This being granted, let a Logarithmetic Curve be ſuppoſed to be deſcribed through the Points Nn, whoſe Axis is OMQ. Then, in this Curve, if the Proportion of the Axis MQ expreſſes the Time, the Ordinate QY will repreſent the Money proportionally increaſing every Moment, to that Time. For if there be taken ml, &c. $= Mm$, the Ordinates lp, &c. ſhall be a Series of continual Proportionals in the Ratio of MN to mn; that is, they increaſe in the ſame Ratio as the Money doth.

Again, Let the Right Line NX touch the Logarithmetical Curve in N, and the Subtangent thereof MX ſhall be conſtant and invariable, and the ſmall Triangle Non ſhall be ſimilar to the Triangle XMN. But it has been proved, that the Increment $no = Mm \times a = No \times a$; and ſo $no : No :: No \times a : No :: a : 1$. But as no is to No, ſo ſhall MN be to MX. Wherefore it ſhall be, as a is to 1, ſo is MN, or 1, to $MX = \frac{1}{a} =$ Subtangent.

Now if the yearly Rate of Intereſt be $\frac{1}{20}$ Part of the Principal, or if $a = \frac{1}{20} = .05$, then will $MX \times \frac{1}{a} = 20$.

Becauſe in different Forms of Logarithms, the Logarithms of the ſame Number are proportional to the Subtangents of their Curves: If MQ expreſſes the Time of a whole Year, or Unity, then ſhall QY be the Amount of the Money at the Year's End. And to find QY, ſay, As MX, or $\frac{1}{20}$, is to 0.4342944 (which Number expounds the Subtangent of the Logarithmetic Curve expreſſing *Briggs*'s Logarithms), ſo is one Year, or Unity, to a *Briggian* Logarithm, anſwering to the Number QY. This Logarithm will be found 0.0217147, and the Number anſwering to the ſame is $1.05127 = QY$, whoſe Increment above

Unity, or the Principal, exceeds the yearly Intereſt ,05 but a ſmall Matter. And ſo if the yearly Intereſt of 100 Pounds, be 5 Pounds, the proportional yearly Intereſt, which is added to the Principal 100 at the End of each Particle of the Year, will amount only at the Year's End to 5 Pounds 2 Shillings and $6\frac{1}{2}$ Pence.

And if ſuch a Rate of Intereſt be required, that every Moment a Part of it continually proportional to the increaſing Principal be added to the Principal, ſo that at the Year's End an Increment be produced that ſhall be any given Part of the Principal; for Example, the $\frac{1}{20}$ Part; ſay, As the Logarithm of the Number 1.05 is to 1; that is, as 0.0211893 is to 1; ſo is the Subtangent 0,4342944 to $\frac{1}{a} = 20.49$, and then will $a = \frac{1}{20.49} = .0488$. For if ſuch a Part of the Rate of Intereſt .0488 be ſuppoſed, as anſwers to a Moment, that is, having the ſame Ratio to .0488 as a Moment has to a Year, and it be made, as Unity is to that Part of the Rate of Intereſt, ſo is the Principal to the momentaneous Increment thereof; then will the Money, continually increaſing in that Manner, be augmented, at the Year's End, the $\frac{1}{20}$ Part thereof.

CHAP. VI.

Of the Method by which Mr. Briggs *computed his Logarithms, and the Demonſtration thereof.*

ALthough Mr. *Briggs* has no where deſcribed the Logarithmetical Curve, yet it is very certain, that, from the Uſe and Contemplation thereof, the Manner and Reaſon of his Calculations will appear. In any Logarithmetical Curve HBD, let there be three Ordinates A B, *a b*, *q s*, nearly equal to one another; that is, let their Differences have a very ſmall Ratio to the ſaid Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For ſince the Ordinates are nearly equal to one another, they will be very nigh to each other; and ſo the Part of the Curve Bs, intercepted by them, will almoſt coincide with a ſtrait Line; for it is certain, that the Ordinates may be ſo near to each other, that the Difference between the Part of the Curve, and the Right Line ſubtending it, may have to that Subtenſe a Ratio leſs than any given Ratio. Therefore the Triangles Bcb, Brs, may be taken for Right-lined, and will be equiangular. Wherefore, as $sr : bc :: Br : Bc :: Aq : Aa$; that is, the Exceſſes of the Ordinates, or Lines above the leaſt, ſhall be proportional to the Differences of their Logarithms. And from hence appears the Reaſon of the Correction of Numbers and Logarithms by Differences and proportional Parts. But if AB be Unity, the Logarithms of Numbers ſhall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10; or, which is the ſame Thing, if the Square Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof ſhall be $\frac{1}{2}$ of the Logarithm of 10; and ſo will be given. If, again, between the Number before found, and Unity, there be found a mean Proportional, which may be done by extracting the Square Root of the ſaid Number, this Number or Root, will be twice nearer to Unity than the former, and its Logarithm will be one Half of the Logarithm of that, or one Fourth of the Logarithm of 10. And if in this manner the Square Root be continually extracted, and the Logarithms biſected, you will at laſt get a Number, whoſe Diſtance from Unity ſhall be leſs than the $\frac{1}{1000000000000000}$ Part of the Logarithm of 10. And after Mr. *Briggs* had made 54 Extractions of the Square Root, he found the Number 1.00000 00000 00000 12781 91493 20032 3442; and its Logarithm was 0. 00000 00000 00000 05551 11512 31257 82702. Suppoſe this Logarithm to be equal to Aq, or Br, and let qs be the Number found by extracting the Square Root; then will the Exceſs of this Number above Unity, *viz.* rs = ,00000 00000 00000 12781 91493 20032 3442.

Now, by means of theſe Numbers, the Logarithms of all other Numbers may be found in the following manner: Between the given Number (whoſe Logarithm is to be found) and Unity, find ſo many mean Proportionals (as above), till at laſt a Number be gotten ſo little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of ſignificative Figures after thoſe. Let this Number be ab, and let the ſignificative Figures, with the Cyphers prefixed before them, denote the Difference bc. Then ſay, As the Difference rs is to the Difference bc, ſo is Br a given Logarithm, to Bc, or Aa, the Logarithm of the Number ab; which therefore is given. And if this Logarithm be continually doubled, the ſame Number of Times as there were Extractions of the Square Root, you will at laſt have the Logarithm of the Number ſought. Alſo, by this Way may the Subtangent of the Logarithmetic Curve be found, *viz.* by ſaying, As $rs : Br :: AB$, or Unity : AT, the Subtangent, which therefore will be found to be 0.434294482903251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards, as alſo its Logarithm, and the Logarithm of another Number, ſufficiently near to NM, be ſought, ſay, As NM is to the Subtangent XM, ſo is no, the Diſtance of the Numbers, to No, the Diſtance of the Logarithms. Now, if NM be Unity $= AB$, the Logarithms will be had by multiplying the ſmall Differences bc by the conſtant Subtangent AT.

By this Way may be found the Logarithms of 2, 3, and 7; and by theſe the Logarithms of 4, 8, 16, 32, 64, &c. 9, 27, 81, 243, &c. as alſo 7, 49, 343, &c. And if from the Logarithm of 10 be taken the Logarithm of 2, there will remain the Logarithm of 5; ſo there will be given the Logarithms of 25, 125, 625, &c.

The Logarithms of Numbers compounded of the aforeſaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, &c. are eaſily had by adding together the Logarithms of the component Numbers.

But ſince it was very tedious and laborious to find the Logarithms of the prime Numbers, and not eaſy to compute Logarithms by Interpolation, by firſt, ſecond, and third, &c. Differences; therefore the great Men, Sir *Iſaac Newton*, *Mercator*, *Gregory*, *Wallis*, and, laſtly, Dr. *Halley*, have publiſhed infinite converging Series, by which the Logarithms of

Numbers

Pag. 356.

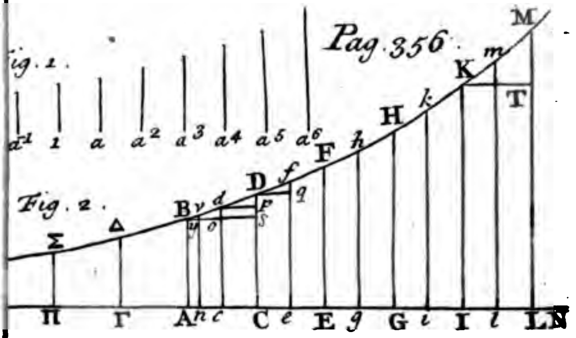
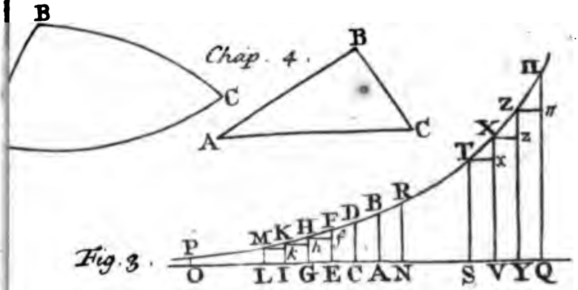

a⁻¹ 1 a a² a³ a⁴ a⁵ a⁶
M m K T k H h F f q D d p s B v y o Δ Σ
Π Γ A n c C e E g G i I l L N

Fig. 1.

Fig. 2.



Chap. 4.

Fig. 3.

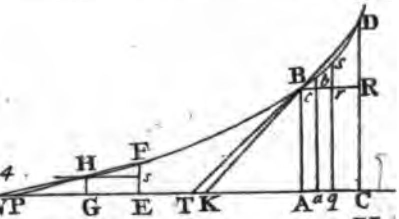


Fig. 4.

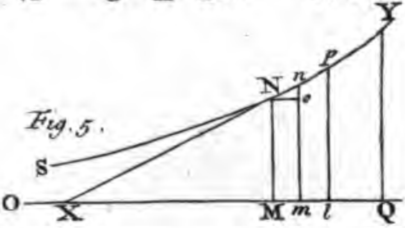


Fig. 5.

OF

Numbers to any Number of Places may be had more expeditiouſly, and truer: Concerning which Series Dr. *Halley* has written a learned Tract, in the *Philoſophical Tranſactions*; wherein he has demonſtrated thoſe Series after a new Way, and ſhews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, eaſily and expeditiouſly, the Logarithms of large Numbers.

Let z be an odd Number, whoſe Logarithm is ſought; then ſhall the Numbers $z-1$ and $z+1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms, will be had, which let be called y. Therefore, alſo, the Logarithm of a Number, which is a Geometrical Mean between $z-1$ and $z+1$, will be given, *viz.* equal to the Half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25220z^9}$, &c. ſhall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z-1$ and $z+1$ has to the Arithmetical Mean, *viz.* to the Number z.

If the Number exceeds 1000, the firſt Term of the Series $\frac{y}{4z}$ is ſufficient for producing the Logarithm to 13 or 14 Places of Figures, and the ſecond Term will give the Logarithm to 20 Places of Figures. But, if z be greater than 10000, the firſt Term will exhibit the Logarithm to 18 Places of Figures; and ſo this Series is of great Uſe in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example, it is required to find the Logarithm of 20001. The Logarithm of 20000 is the ſame as the Logarithm of 2 with the Index 4 prefixed to it; and the Difference of the Logarithms of 20000 and 20002 is the ſame as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.00004 34272 7687. And if this Difference be divided by $4z$, or 80004, the Quotient $\frac{y}{4z}$

ſhall be - - - - - - - 0.00000 00005 42814
And if the Logarithm of the 4.30105 17093 02416
Geometrical Mean be added 4.30105 17098 45230
to the Quotient, the Sum will
be the Logarithm of 20001. Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond ſix Places of Figures. But if you have a mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Table, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will moſtly conſiſt in ſetting down the Numbers.

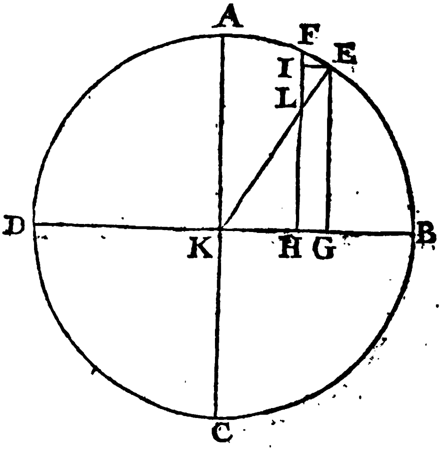
Note, This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a mind to be informed more in this Matter, let them conſult his above-named Treatiſe.

THE

APPENDIX.

IT is needleſs here to write a Prefatory Diſcourſe, ſetting forth the Uſe and Invention of Logarithms, ſince the Author has ſupplied that, in his Preface, to the Treatiſe of the Nature and Arithmetic of Logarithms annexed to theſe Elements: It is enough to inform the Reader, that my chief Deſign in writing this *Appendix* was, to render their Conſtruction eaſy, by inveſting various Theorems for that Purpoſe, and illuſtrating them by proper Examples; all which is performed in the actual Operation of making the Logarithms of the firſt 10 Numbers, and of the prime Number 101, which is more than ſufficient to inform the meaneſt Capacity how to examine or conſtruct the whole Table. I have alſo ſhewn how, from the Logarithm given, to find its correſponding Number; and the Inveſtigation of the Series omitted by the Author in Page 357, for expeditiouſly finding the Logarithms of large Numbers. As to thoſe Series exhibited by him in his Trigonometrical Treatiſe, Page 287, for making the Sines and Coſines; I muſt declare, that I have exceeded my firſt Intentions, which were to give their Inveſtigation only; but conſidering, that as they depended upon the *Newtonian* Series without the Inveſtigation of which our Author's Series could never be thoroughly underſtood; I thought it would therefore prove acceptable, if I ſhewed their Inveſtigations too, from which thoſe of our Author eaſily flow. In order to which, and to keep the Reader no longer in Suſpenſe, let r be put for the Radius K E of the Circle A B C D; a for the Arc B E, whoſe Length is to be inveſtigated; and s equal to the Sine E G: Then is F E $= a$, and I F $= s$.

Then,



Then, the Triangles K G E, K H L, are ſimilar, becauſe L H is parallel to E G; the Triangles K L H F L E, are ſimilar, becauſe the Angles K L H, F L E, are equal, and the Angles K H L, F E L, are both Right Angles; and the Triangles F L E, F E I, are ſimilar, becauſe the Angles F E L, F I E, are both Right Angles, and the Angle F is common: Therefore the Triangles F I E and K G E are ſimilar, whence, KG (or $\sqrt{rr-ss}$) : KE (or r) : : FI (or $\dot{s}$) : F E (or $\dot{a}$); that is, $\dot{a}=\frac{r\dot{s}}{\sqrt{rr-ss}}$.

Now $\sqrt{rr-ss}=r-\frac{ss}{2r}-\frac{s^4}{8r^3}-\frac{s^6}{16r^5}-\frac{s^8}{128r^7}$, &c. by extracting the Square Root:

And if $r\dot{s}$ be divided by that Series, the Quotient, $\dot{s}+\frac{s^2\dot{s}}{2rr}+\frac{3s^4\dot{s}}{8r^4}+\frac{5s^6\dot{s}}{16r^6}+\frac{35s^8\dot{s}}{128r^8}$, &c. will be the Fluxion of the Arc; therefore the Fluent thereof, *viz.* $\left(s+\frac{s^3}{2.3rr}+\frac{3s^5}{2.4.5r^5}+\frac{5s^7}{2.4.2.7r^6}+\frac{35s^9}{2.4.2.8.9r^8}\right.$, &c. or$\left.\right)$ $s+\frac{s^3}{2.3r^2}+\frac{3.3s^5}{2.3.4.5r^4}+\frac{3.3.5.5s^7}{2.3.4.5.6.7r^6}+\frac{3.3.5.5.7.7s^9}{2.3.4.5.6.7.8.9r^8}$ &

&c. will be equal to the Arc of a Circle whoſe Radius is r, and Sine s. But if r be put equal to Unity, then $s + \frac{1}{2.3}s^3 + \frac{3.3}{2.3.4.5}s^5 + \frac{3.3.5.5}{2.3.4.5.6.7}s^7$, &c. will expreſs the Length of the Arc a.

EXAMPLE.

Let it be required to find the Length of the Arc of 30 Degrees to 6 Places of Decimals, the Radius being Unity.

Here $s = \frac{1}{2}$, and $ss = \frac{1}{4}$; whence the Operation may be as follows;

s	$=$,5000000	s	$=$,5000000
s^3	$=$,1250000	$\frac{s^3}{6}$	$=$,208333
s^5	$=$ 312500	$\frac{3s^5}{40}$	$=$,23437
s^7	$=$ 78125	$\frac{5s^7}{112}$	$=$,3487
s^9	$=$ 19531	$\frac{35s^9}{1152}$	$=$,593
s^{11}	$=$ 4882	$\frac{63s^{11}}{2816}$	$=$,109
s^{13}	$=$ 1220	$\frac{231s^{13}}{13312}$	$=$,21
s^{15}	$=$ 305	$\frac{143s^{15}}{10240}$	$=$,4
			,5235984

Hence the Length of the Arc of 30 Degrees is ,523598+. Now if this Arc be multiplied by 6, we ſhall have the Length of the Arc of the Semicircle in ſuch Parts as the Radius is 1, or of the whole Circumference in ſuch Parts as the Diameter is 1, *viz.* 3,14159+.

But there is no Series ſo eaſy to be retained in the Memory, and ſo readily put in Practice, for obtaining the Ratio of the Diameter of the Circle to its Circumference, as that which is derived from the Tangent. For if t be put equal to the Tangent of any Arc, then $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9$, &c.

Now the Radius being Unity the Sine of 30 Degrees $= \frac{1}{2}$, and conſequently the Coſine $= \sqrt{\frac{3}{4}}$; and becauſe the Coſine is to the Right Sine, as the Radius to the Tangent; it will be $\sqrt{\frac{3}{4}} : \sqrt{\frac{1}{4}} :: 1 : \sqrt{\frac{1}{3}}$, the Tangent of 30° 00′ $= t$, whence $tt = \frac{1}{3}$. Wherefore, if the Root of $\frac{1}{3}$ be divided continually by 3 ſeveral Quotients by the odd Numbers ſucceſſively, *viz.* the firſt by 3, the ſecond by 5, *&c.* the Sum of the affirmative Quotients made leſs by all the negative ones, will be the Arc of 30 Degrees.

And becauſe the Arc of 30 Degrees is $\frac{1}{6}$ Part of the Semicircumference, if inſtead of $\sqrt{\frac{1}{3}}$ be taken $6\sqrt{\frac{1}{3}} = \sqrt{12}$, we ſhall have the Semicircumference in ſuch Parts as the Radius is Unity; or the whole Circumference, the Diameter being Unity.

The OPERATION ſtands thus:

$\sqrt{12} =$	3,464102	+	3,464102		
3)	1,154701(			—	,384900
5)	,384900(	+	76980		
7)	,128300(			—	18329
9)	42767(	+	4752		
11)	14256(			—	1296
13)	4752(	+	366		
15)	1584(			—	106
17)	528(	+	31		
19)	176(			—	9
		+	1,546231	—	,404640

Whence 3,546231 —,404640 = 3,141591 the ſame as before. The Impoſſibility of expreſſing the exact Proportion of the Diameter of a Circle to its Circumference, by any received Way of Notation, has put the moſt celebrated Men, in all Ages, upon approximating the Truth as near as poſſible; there being a Neceſſity of a near Quadrature, inaſmuch as it is the Baſis upon which the moſt uſeful Branches of the Mathematics are built. And after the famous *Van Ceulen*, who carried it to 36 Places of Decimals (which he order'd to be engraven on his Tomb ſtone, thinking he had ſet Bounds to farther Improvements), the firſt that attempted it with Succeſs was the moſt indefatigable Mr. *Abraham Sharp*, who, by a double Computation, *viz.* from the Sine of 6 Degrees one Way, and from the Sine and Coſine of

12 Degrees another Way, carried it to twice the Number of Places that *Van Ceulen* had done, *viz.* 72.

And ſince that time Mr. *Machin*, late Profeſſor of Aſtronomy in *Greſham College*, and Secretary to the *Royal Society*, by a different Method of Computation, has carried it to 100 Places, almoſt triple the Number that *Van Ceulen* had done, which not only confirms Mr. *Sharp*'s Quadrature, but ſhews us, that, if the Diameter be 1,00000, *&c.* the Circumference will be 3,14159. 26535. 89793. 23846. 26433. 83279. 50288. 41971. 69399. 37510. 58209. 74944. 59230. 78164. 06286. 20899. 86280. 34825. 34211. 70680 + of the ſame Parts.

Which is a Degree of Exactneſs far ſurpaſſing all Imagination; being, by Eſtimation, more than ſufficient to calculate the Number of Grains of Sand that may be comprehended within the Sphere of the Fixed Stars.

The late Mr. *Cunn*'s Series for determining the Periphery of an Ellipſis (who was my Predeceſſor in the Mathematical School, erected by *Frederic Slare*, M. D. and eſtabliſhed by a Decree of the High Court of Chancery for qualifying Boys for the Sea-Service) being new and curious, this Opportunity is taken of making it public.

Let A be equal to a Quadrant of the Circle circumſcribing the Ellipſis, whoſe Periphery is required. Then $A \times$

$$1 - \frac{1.1}{2.2}e^2 - \frac{1.3.1}{2.4.8}e^4 - \frac{1.3.5.1}{2.4.6.16}e^6 - \frac{1.3.5.7.5.}{2.4.6.8.128}e^8 - \frac{1.3.5.7.9.7}{2.4.6.8.10.256}e^{10} - \frac{1.3.5.7.9.11.21}{2.4.6.8.10.12.1024}e^{12}, \text{ \&c.}$$

is the Periphery of a Quadrant of the Ellipſis, where $ee = \frac{tt - cc}{tt}$, t being the Semi-tranſverſe Diameter, and c the Semi-conjugate.

When this Series came to hand, it was imperfect, inaſmuch as there were only the firſt five Terms without the Law of Continuation: But, being deſirous of rendering it compleat, after ſome Conſideration, I found the Law to be as follows: It is plain by Inſpection, that the Numerators and Denominators of each Term are compos'd of Numbers that run in arithmetical Progreſſion, except the laſt in each Term, *viz.* $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{5}{128}$, *&c.* and thoſe being found by the continual Multiplication of theſe Fractions, $\frac{1}{1}\times\frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}\times\frac{5}{8}\times\frac{7}{10}\times\frac{9}{12}$, *&c.* the Law of continuing the whole Series as above, is evident. Whence, by a well known Method of ſubſtituting Capital Letters for each *Term reſpectively, the following Series is deduced,* viz. $A\times 1-\frac{1}{4}ee-\frac{1.3}{4.4}e^2B-\frac{3.5}{6.6}e^2C-\frac{5.7}{8.8}e^2D-\frac{7.9}{10.10}e^2E-\frac{9.11}{12.12}e^2F$, *&c.* where the Law of Continuation is evident alſo, ſince each Capital Letter is equal to its precedent Term, *viz.* $B=\frac{1}{4}ee$, $C=\frac{1.3}{4.4}e^2B$, *&c.* and without doubt in Practice is preferable to the former Series: But the Inveſtigation of that, on which this laſt depends, is omitted; purely on account of its being foreign to the preſent Subject.

But to return; if the Series expreſſing the Length of the Arc, *viz.* $s+\frac{1}{6}s^3+\frac{3}{40}s^5$, *&c.* be reverſed, we ſhall have the Value of s in the Terms of a, and conſequently a direct Method for finding the Sine of any Arc from its Length given. Thus,

If $a=s+\frac{1}{6}s^3+\frac{3}{40}s^5+\frac{5}{112}s^7$, *&c.*

Then $s=a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{040}a^7$, *&c.*

Or $s=a-\frac{a^3}{2.3}+\frac{a^5}{2.3.4.5}-\frac{a^7}{2.3.4.5.6.7}$, *&c.*

$$\left.\begin{array}{ll}\text{For put } s= & Aa+Ba^3+Ca^5,\ \&c.\\ \text{Then } \frac{1}{6}s^3= & \frac{1}{6}A^3a^3+\frac{1}{2}A^2Ba,\ \&c.\\ \text{And } \frac{3}{40}s^5= & +\frac{3}{40}A^5a^5\ \&c.\end{array}\right\}=a.$$

And conſequently $Aa=a$; whence $A=1$:

Alſo, $B+\frac{1}{6}A^3=0$; or $B=(-\frac{1}{6}A^3=)-\frac{1}{6}$;

Again $C+\frac{1}{2}A^2B+\frac{3}{40}A^5=0$; or $C=-\frac{1}{2}A^2B-\frac{3}{40}A^5$;

That is $C=(\frac{1}{2}\times 1\times\frac{1}{6}-\frac{3}{40}=\frac{10}{120}-\frac{9}{120}=)\frac{1}{120}$

Wherefore $A=1$, $B=-\frac{1}{6}$, $C=\frac{1}{120}$, *&c.* and conſequently, $s=a-\frac{a^3}{6}+\frac{a^5}{120}$, *&c.* From which three Terms the Law of Continuation is eaſily diſcovered: Therefore $s=a-\frac{a^3}{2.3}+\frac{a^5}{2.3.4.5}-\frac{a^7}{2.3.4.5.6.7}+\frac{a^9}{2.3.4.5.6.7.8.9}$, *&c.*

Whence, ſubſtituting A for a, and we ſhall have $A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} \quad \frac{A}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9}$, &c. for the *Newtonian* Series, according to our Author's Form, for finding the Sine of any Arc, its Length being given. Q. E. I.

Again, becauſe the Square of the Radius, made leſs by the Square of the Sine, is equal to the Square of the Coſine, by the ſecond Propoſition of our Author's Elements of Plane Trigonometry; it follows, that if from the Square of the Radius $= 1$ be taken the Square of the Sine $= a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the Square Root of the Remainder will be the Coſine $= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6$, &c. Thus,

$$\begin{array}{rl}
s = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\
s = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\
\hline
 & a^2 - \frac{1}{6}a^4 + \frac{1}{120}a^6, \text{ \&c.} \\
 & \quad - \frac{1}{6}a^4 + \frac{1}{36}a^6, \text{ \&c.} \\
 & \qquad\qquad + \frac{1}{120}a^6, \text{ \&c.} \\
\hline
ss = & a^2 - \frac{1}{3}a^4 + \frac{2}{45}a^6, \text{ \&c.}
\end{array}$$

which, being taken from the Squares of the Radius 1, leaves $1 - aa + \frac{1}{3}a^4 - \frac{2}{45}a^6$, &c. the Square Root of which will be the Coſine.

$$1 - aa + \frac{1}{3}a^4 - \frac{2}{45}a^6, \text{ \&c.} \left(\begin{array}{l} 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 \\ - \frac{1}{720}a^6, \text{ \&c.} \end{array} \right.$$

$$\begin{array}{rl}
 & 1 \\
\hline
2 - \frac{1}{2}aa) & -aa + \frac{1}{3}a^4 \\
 & -aa + \frac{1}{4}a^4 \\
\hline
2 - aa, \text{ \&c.}) & \frac{1}{12}a^4 - \frac{2}{45}a^6, \text{ \&c.} \\
 & \frac{1}{12}a^4 - \frac{1}{24}a^6, \text{ \&c.} \\
\hline
2 - aa, \text{ \&c.}) & \quad - \frac{1}{363}a^6, \text{ \&c.} \\
 & \quad - \frac{1}{360}a^6, \text{ \&c.} \\
\hline
 & \qquad 0
\end{array}$$

Wherefore putting A for a, we ſhall have the Coſine $1 - \frac{A^2}{2} + \frac{A^4}{24} - \frac{A^6}{720}$, &c. or $1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8}$, &c. according to the Author's Form. Q. E. I.

But becauſe thoſe Series, as our Author obſerves, converge very ſlowly, eſpecially when the Arc is nearly equal to the Radius; he therefore deviſed (Page 278.) other Series, whoſe Inveſtigation may be as follows:

Let the Arc, whoſe Sine is ſought, be the Sum or Difference of two Arcs, *viz.* A+z, or A—z; and let the Sine of the Arc A be called *a*, and the Coſine *b*.

Now, if the Arc DF=DE, *Prop. 5th of the Elements of Trigonometry*, be called z, then its Sine FO will, by the *Newtonian* Series, be $=z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2.3.4.5}-\frac{z^7}{1.2.3.4.5.6.7}$, &c. and its Coſine CO $=1-\frac{z^2}{1.2}+\frac{z^4}{1.2.3.4}-\frac{z^6}{1.2.3.4.5.6}$, &c. and becauſe CD : DK : : CO : OP; therefore OP $=a-\frac{az^2}{1.2}+\frac{az^4}{1.2.3.4}-\frac{az^6}{1.2.3.4.5.6}$, &c.

Again, becauſe the Triangles CDK, FOM, are ſimilar, it will be, as CD : CK : : FO : FM; whence FM $=\frac{bz}{1}-\frac{bz^3}{1.2.3}+\frac{bz^5}{1.2.3.4.5}-\frac{bz^7}{1.2.3.4.5.6.7}$, &c.

But OP+FM=IF, the Sine of the Arc BF, *viz.* A+z; conſequently the Sum of thoſe Series, *viz.* $a+\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}$, &c. is the Sine of the Arc A+z. And becauſe FM=MC, therefore their Difference $a-\frac{bz}{1}-\frac{az^2}{1.2}+\frac{bz}{1.2.3}+\frac{az^4}{1.2.3.4}-\frac{bz^5}{1.2.3.4.5}$, &c. is the Sine of the Arc A—z, *viz.* EL.

And again, becauſe CD : CK : : CO : CP; therefore CP $=b-\frac{bz^2}{1.2}+\frac{bz^4}{1.2.3.4}-\frac{bz^6}{1.2.3.4.5.6}+\frac{bz^8}{1.2.3.4.5.6.7.8}$, &c. and by reaſon of the ſimilar Triangles CDK, FMO, it will be, as CD : DK : : FO : MO. Whence MO $=az-\frac{az^3}{1.2.3}+\frac{az^5}{1.2.3.4.5}-\frac{az^7}{1.2.3.4.5.6.7}$, &c.

[illegible]

But CP—MO=CI, the Coſine of the Arc A+z. Wherefore the Coſine of the Arc A + z is $b - \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6}$ &c.

And becauſe MO = PL, therefore CP + MO = CL, and conſequently the Coſine of the Arc A—z = $b + \frac{az}{1} - \frac{bz^2}{1.2} - \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5}$, &c. Q. E. I.

Now the Arc A is an Arithmetical Mean between the Arcs A—z and A+z, and the Difference of their Sines are $\frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6}$, &c. $\frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6}$, &c. Whence the Difference of the Differences, or ſecond Difference, is $\frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6}$, &c. or $2a \times \overline{\frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6}}$, &c. Which Series is equal to double the Sine of the mean Arc, drawn into the verſed Sine of the Arc z, and converges very ſoon; ſo that if z be the Arc of the firſt Minute of the Quadrant, our Author ſays the firſt Term of the Series gives the ſecond Difference to 15 Places of Figures, and the ſecond Term to 25 Places.

Whence the following Rule is derived for finding the Sine of the Arc A+z, or A—z.

RULE.

From double the Sine of the mean or middle Arc, ſubtract the ſecond Difference found by the Theorem, and from the Remainder ſubtract the Sine of the given Extreme, whether it be the greater or leaſt; and the Remainder will be the Sine of the other Extreme.

EXAMPLE.

Let it be required to find the Sine of 30° 01′; the Sines of 30° 00′, and 29° 59′, being both given.

Here 30° 00′ is the mean Arc, whoſe Sine is ,500000 00000; and the Sine of 29° 59′, the given Extreme, is ,49974806226, and the Length of the Arc z, *viz.* one Minute, is ,000290888208; which ſquared and multiplied by the Sine of the mean Arc 50000, *&c.* according to the Direction of the Theorem, the Product will be the ſecond Difference, equal to ,000000042307; which ſubtracted from double the Sine of the mean Arc, equal to 1, the Remainder will be ,999999957693; from which ſubtract the Sine of the given Extreme (which in this Caſe is the leaſt), and there will remain ,50025189543 for the Sine of 30° 01′, the greater Extreme.

This Method of making the Sines, however it may appear at firſt Sight, is ſo far from being tedious or troubleſome, that I look upon it to be the moſt eligible of any other whatſoever: For the Square of z being once determined, and the ſeveral Multiples of it by the nine Digits made, and ſet down in a Table orderly, all the Sines may be made by Addition and Subtraction only; as indeed our Author hints they may by the Method demonſtrated in the tenth Propoſition of the Elements of Trigonometry; but this is evidently preferable to that, tho' a good Method too; and by which all the Sines of the Quadrant, I preſume, were wont to be made, at leaſt as far as 30, or 60, Degrees; for after the Sines as far as 60 Degrees are obtained, all the others may be had by Addition only; and notwithſtanding there are other excellent Theorems, which contribute very much towards finiſhing and confirming the Truth of the whole Canon; yet this deduced from our Author's Series, I deem the moſt elegant and fit for Practice, becauſe the Difference of the Differences of the Sines being what is always required to be found, there will be ſeven Cyphers, at leaſt, before the ſignificant Figures of the ſaid Difference; which is the Product made by the Square of z, into the Sine of the mean Arc: So that to have the Sine true to ten Places, there will not be occaſion to find above four or five Figures in the Product, which, according to the common Method of contracted Multiplication, may be obtained with very few Figures. Thus, for Inſtance, the Sine of 30° 02′ may be had to ten Places by a wonderful eaſy Operation; the Sines of 30° 01′ and 30 00 being both given.

EXAMPLE.

The Sine of 30° 01′ is	,50025189543
The Square of z inverted	26480000000
	40020
	2001
	300
	10
	42331

Whence the Product is ,000000042331 true to eleven Places at leaſt. Wherefore if, according to the Rule, from double the Sine of the middle Arc = 1,00050379086 we ſubtract the ſaid Product, ,00000004233

double the Sine of the middle Arc =	1,00050379086
we ſubtract the ſaid Product,	,00000004233
And from the Remainder	1,00050374853
the Sine of 30° 00′ the given Extreme	,50000000000
be ſubtracted	,50050374853

There will remain ,50050374853 for the Sine of 30° 02′ the other Extreme; than which, nothing of this Nature can be deſired more eaſy.

SCHOLIUM.

Becauſe the Difference of the Differences of the Sines, or ſecond Difference, has always 7 Cyphers before the ſignificant Figures; it follows, that the whole Canon, where the Sines conſiſt but of 6 Places, which is as far as our Tables for common Practice need extend, may be performed chiefly by Addition and Subtraction only, without forming Multiples of the Square of z by the nine Digits; tho' perhaps it may be neceſſary to uſe the Method of contracted Multiplication every 5th Minute, to confirm the Truth, leſt, in continual doubling and ſubtracting, an Error ſhould ariſe in the Right-hand Figure: However, as it may be ſafely uſed for 5 Minutes together, and ſometimes more, it will render the Whole very eaſy.

Note, The Square of z in this Caſe, *viz.* the Arc of 5 Minutes, is ,000002115.

Thus having inveſtigated the *Newtonian* and our Author's Series, and exemplified the latter, by making the Sines of 30° 01′ and 30° 02′, and withal ſhewn how, from the Sine of the Arc given, to find the Length of that Arc, and conſequently the Circumference of the whole Circle; I ſhall beg Leave, before I treat of the Conſtruction of Logarithms, to ſhew how, from the known Ratio of the Diameter to the Circumference, or any other Ratio whatſoever, that a Set of integral Numbers may be found, whoſe Ratios ſhall be the neareſt poſſible to the Ratio given; for which I hope to be excuſed, and the rather, becauſe I believe this Method of determining them was never before publiſhed.

RULE.

Divide the Conſequent by the Antecedent, and the Diviſor by the Remainder, and the laſt Diviſor by the laſt Remainder, and ſo on till nothing remains.

Then for the Terms of the firſt Ratio, Unity will always be the Antecedent, and the firſt Quotient the firſt Conſequent.

For the TERMS of the ſecond RATIO.

Multiply the laſt $\left\{\begin{matrix}\text{Antecedent}\\ \text{Conſequent}\end{matrix}\right\}$ by the 2d Quotient; and to the Product add $\left\{\begin{matrix}\text{Nothing;}\\ \text{Unity;}\end{matrix}\right\}$ and ſo will the Reſult be the ſecond $\left\{\begin{matrix}\text{Antecedent.}\\ \text{Conſequent.}\end{matrix}\right.$

For all the following RATIOS:

Multiply the laſt $\left\{\begin{matrix}\text{Antecedent}\\ \text{Conſequent}\end{matrix}\right\}$ by the next Quotient, and to the Product add the laſt $\left\{\begin{matrix}\text{Antecedent}\\ \text{Conſequent}\end{matrix}\right\}$ but one; and ſo will the Sum be the preſent $\left\{\begin{matrix}\text{Antecedent.}\\ \text{Conſequent.}\end{matrix}\right.$

EXAMPLE.

Let it be required to find a Rank of Ratios, whoſe Terms are integral, and the neareſt poſſible to the following Ratio, *viz.* of 10000 to 31416, which expreſſes nearly the Proportion of the Diameter of the Circle to its Circumference.

But becauſe the Terms of the Ratio are not prime to each other, they muſt therefore be reduced to their leaſt Terms.

Whence $\frac{10000}{31416}=\frac{1250}{3927}$, and then 3927 divided by 1250, and 1250 by the Remainder, &c. will be as follows:

$$\begin{array}{llllllll} 1250\ ) & 3927\ (& 3 & & & & & \\ & 177\ ) & 1250\ (& 7 & & & & \\ & & 11\ ) & 177\ (& 16 & & & \\ & & & 1\ ) & 11\ (& 11 & & \\ & & & & 0 & & & \end{array}$$

So the firſt Antecedent is 1, and the firſt Conſequent 3.

$$\text{Let}\left\{\begin{array}{ll}\text{Anteced.} & 1\\ \text{Conſeq.} & 3\end{array}\right\}\times 7=\left\{\begin{array}{c}7\\ 21\end{array}\right\}\text{And}\left\{\begin{array}{l}7+0=7\text{ the ſecond Antec.}\\ 21+1=22\text{ the ſecond Conſ.}\end{array}\right.$$

Which 7 and 22 is *Archimedes*'s Proportion.

$$\text{Let}\left\{\begin{array}{ll}\text{Anteced.} & 7\\ \text{Conſeq.} & 22\end{array}\right\}\times 16=\left\{\begin{array}{c}112\\ 352\end{array}\right\}\text{And}\left\{\begin{array}{l}112+1=113\text{ the 3d Ant.}\\ 352+3=355\text{ the 3d Conſ.}\end{array}\right.$$

Which Terms 113 and 355 is *Metius*'s Proportion.

$$\text{Let}\left\{\begin{array}{ll}\text{Antecedent} & 113\\ \text{Conſequent} & 355\end{array}\right\}\times 11=\left\{\begin{array}{c}1243\\ 3905\end{array}\right\}\text{And}\left\{\begin{array}{l}1243+7=1250\\ 3905+22=3927\end{array}\right.$$

Producing the ſame Antecedent and Conſequent as at firſt; which, as it is ever the Property of the Rule ſo to do, proves, at the ſame Time, that no Error has been committed thro' the whole Operation.

$$\text{Whence, as }1250:3927::\left\{\begin{array}{c}1:3\\ 7:22\\ 113:355\end{array}\right\}\begin{array}{l}\text{For the}\\ \text{Terms}\\ \text{of the}\end{array}\left\{\begin{array}{c}1\\ 2\\ 3\end{array}\right\}\text{Ratio.}$$

But it muſt be obſerved, that 1 to 3 does not expreſs the Ratio ſo near as 7 to 22; nor 7 to 22 ſo near as 113 to 355; that is, the larger the Terms of the Ratio are, the nearer they approach the Ratio given.

Mr. *Molyneux*, in his Treatiſe of Dioptrics, informs us, that when Sir *Iſaac Newton* ſet about, by Experiments, to determine the Ratio of the Angle of Incidence, to the refracted Angle, by the means of their reſpective Sines; he found it to be, from Air to Glaſs, as 300 to 193, or, in the leaſt round Numbers, as 14 to 9. Now, if it be as 300 is to 193, it will readily appear, by the Rule, whether they are ſuch integral Numbers, whoſe Ratio is the neareſt poſſible to the given Ratio.

$$\begin{array}{l} 193\ )\ 300\ (\ 1 \\ \qquad 107\ )\ 193\ (\ 1 \\ \qquad\qquad 86\ )\ 107\ (\ 1 \\ \qquad\qquad\qquad 21\ )\ 86\ (\ 4 \\ \qquad\qquad\qquad\qquad 2\ )\ 21\ (\ 10 \\ \qquad\qquad\qquad\qquad\qquad 1\ )\ 2\ (\ 2 \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

For, dividing the great Number by the leſs, and the leſs by the Remainder, &c. the Operation will ſhow that the Numbers 193 and 300 are prime to each other; and that the firſt Antecedent is 1, as alſo the firſt Conſequent.

$$\text{Whence}\left\{\begin{matrix}1\\1\end{matrix}\right\}\times 1=\left\{\begin{matrix}1\\1\end{matrix}\right.\text{And}\left\{\begin{matrix}1+0=1\text{ the 2d Ant.}\\1+1=2\text{ the 2d Con.}\end{matrix}\right.$$

$$\text{Again}\left\{\begin{matrix}1\\2\end{matrix}\right\}\times 1=\left\{\begin{matrix}1\\2\end{matrix}\right.\text{And}\left\{\begin{matrix}1+1=2\text{ the 3d Ant.}\\1+2=3\text{ the 3d Con.}\end{matrix}\right.$$

$$\text{Again}\left\{\begin{matrix}2\\3\end{matrix}\right\}\times 4=\left\{\begin{matrix}8\\12\end{matrix}\right.\text{And}\left\{\begin{matrix}8+1=9\text{ the 4th Ant.}\\12+2=14\text{ the 4th Con.}\end{matrix}\right.$$

Hence, the fourth Antecedent and Conſequent make the Ratio to be as 9 to 14, or, inverſly, as 14 to 9; which not only agrees with Mr. *Molyneux*, but at the ſame Time diſcovers, that they are nearer to the given Ratio, than any other integral Numbers leſs than 92 and 143; which are the neareſt of all to the given Ratio, as will appear by repeating the Proceſs, according to the Direction of the Rule.

Sir *Iſaac Newton* himſelf determines the Ratio out of Air into Glaſs to be as 17 to 11; but then he ſpeaks of the red Light. For that great Philoſopher, in his Differ-

Diſſertations concerning Light and Colours, publiſhed in the *Philoſophical Tranſactions*, has at large demonſtrated, as alſo in his *Optics*, that the Rays of Light are not all homogeneous, or of the ſame Sort, but of different Forms and Figures, ſo that ſome are more refracted than others, tho' they have the ſame or equal Inclinations on the Glaſs: Whence there can be no conſtant Proportion ſettled between the Sines of the Angles of Incidence, and of the refracted Angles.

But the Proportion that comes neareſt Truth, for the middle or green-making Rays of Light, it ſeems, is nearly as 300 to 193, or 14 to 9. In Light of other Colours the Sines have other Proportions. But the Difference is ſo little, that it need ſeldom to be regarded, and either of thoſe mentioned for the moſt Part is ſufficient for Practice. However, I muſt obſerve, that the Notice here taken either of the one or the other, is more to illuſtrate the Rule, and ſhew, as Occaſion requires, how to expreſs any given Ratio in ſmaller Terms, and the neareſt poſſible, with more Eaſe and Certainty, than any Deſign in the leaſt of touching upon Optics.

Wherefore, leſt this ſmall Digreſſion from the Subject in hand, and indeed even from my firſt Intentions, ſhould tire the Reader's Patience, I ſhall not preſume more, but immediately proceed to the Conſtruction of Logarithms.

Of the Conſtruction of Logarithms.

THE Nature of which, tho' our Author has ſufficiently explained in the Deſcription of the Logarithmetic Curve; yet, before we attempt their Conſtruction, it will be neceſſary to premiſe:

That the Logarithm of any Number is the Exponent or Value of the Ratio of Unity to that Number; wherein we conſider Ratio, quite different from that laid down in the fifth Definition of the 5th Book of theſe Elements; for, beginning with the Ratio of Equality, we ſay 1 to 1=0; whereas, according to the ſaid Definition, the Ratio of 1 to 1=1; and conſequently the Ratio here mentioned is of a peculiar Nature, being affirmative when increaſing, as of Unity to a greater Number; but negative when decreaſing. And as the Value of the Ratio of Unity to any Number is the Logarithm of the Ratio of Unity to that Number, ſo each Ratio is ſuppoſed to be meaſured by the Number of equal Ratiunculæ contained between the two Terms thereof: Whence, if in a continued Scale of mean Proportionals, infinite in Number, there be aſſumed an infinite Number of ſuch Ratiunculæ, between any two Terms in the ſame Scale; then that infinite Number of Ratiunculæ is to another infinite Number of the like and equal Ratiunculæ between any other two Terms, as the Logarithm of the one Ratio is to the Logarithm of the other.

But if, inſtead of ſuppoſing the Logarithms compoſed of a Number of equal Ratiunculæ proportionable to each Ratio, we ſhall take the Ratio of Unity to any Number to conſiſt always of the ſame infinite Number of Ratiunculæ, their Magnitudes in this Caſe will be as their Number in the former. Wherefore, if between Unity, and any two Numbers propoſed, there be taken any Infinity of mean Proportionals, the infinitely little Augments or Decrements of the firſt of thoſe Means in each from Unity will be Ratiunculæ; that is, they will be the Fluxions of the Ratio of Unity to the ſaid Numbers; and becauſe the Number of Ratiunculæ in both are equal, their reſpective Sums, or whole Ratios, will be to each other as their Moments or Fluxions; that is, the Logarithm of each Ratio will be as the Fluxion thereof. Conſequently, if the Root of any infinite Power be extracted out of any Number, the Difference of the ſaid Root from Unity ſhall be as the Logarithm of that Number. So that Logarithms, thus produced, may be of as many Forms as we pleaſe to aſſume infinite Indices of the Power whoſe Root we ſeek. As, if the Index be ſuppoſed 100000, &c. we ſhall have the Logarithms invented by *Neper*; but if the ſaid Index be 230258, &c. thoſe of Mr. *Briggs* will be produced.

Wherefore, if $1+x$ be any Number whatſoever, and n infinite, then its Logarithm will be as $\overline{1+x}^{\frac{1}{n}}-1=\frac{1}{n}\times\overline{x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}}$, &c. For the infinite Root of $1+x$ without its Unciæ or prefixed Numbers,

bers, is $1+x+xx+xxx+xxxx$, &c. and the celebrated binomial Theorem invented by Sir *Iſaac Newton* for determining them is $1\times\frac{1}{n}\times\frac{\frac{1}{n}-1}{2}\times\frac{\frac{1}{n}-2}{3}\times\frac{\frac{1}{n}-3}{4}$ &c. or in this Caſe rather $1\times\frac{1}{n}\times\frac{-1}{2}\times\frac{-2}{3}\times\frac{-3}{4}\times\frac{-4}{5}$ &c. for $\frac{1}{n}$ being an Infiniteſimal, is rejected; whence the infinite Root of $\overline{1+x}^{\frac{1}{n}}=1+\frac{x}{n}-\frac{x^2}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. and the Exceſs thereof above Unity, *viz.* $\frac{x}{n}-\frac{x^2}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. is the Augment of the firſt of the mean Proportionals between Unity and $1+x$, which therefore will be as the Logarithm of the Ratio of 1 to $1+x$, or as the Logarithm of $1+x$. But as $\overline{1+x}^{\frac{1}{n}}-1$ is a Ratiuncula, it muſt be multiplied by 10000, &c. infinitely, which will reduce it to Terms fit for Practice, makeing the Logarithm of the Ratio of 1 to $1+x=\frac{1000,\ \&c.}{n}\times\overline{\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}}$, &c. whence if the Index n be taken 1000, &c. as in *Neper*'s Form, the Logarithms will be ſimply $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$, &c. But as n may be taken at Pleaſure, the ſeveral Scales of Logarithms to ſuch Indices will be as $\frac{1000,\ \&c.}{n}$, or reciprocally as their Indices.

Again, if the Logarithm of a decreaſing Ratio be ſought, the infinite Root of $1-x=\overline{1-x}^{\frac{1}{n}}$ will be found by the like Method to be $1-\frac{x}{n}-\frac{x^2}{2n}-\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. which ſubtract from Unity, and the Decrement of the firſt of the infinite Number of Proportionals will appear to be $\frac{1}{n}\times\overline{\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}}$, &c.

which expreſſes the Logarithm of the Ratio of 1 to 1—x, or the Logarithm of 1—x, according to *Neper's* Form, if the Index n be put =10000, *&c.* as before.

And to find the Logarithm of the Ratio of any two Terms, a the leſſer, and b the greater, it will be as $a : b :: 1 : 1+x$; whence $1+x=\frac{b}{a}$; and $x=\left(\frac{b-a}{a}, \text{ or}\right)$ the Difference divided by the leſſer Term when 'tis an increaſing Ratio, and $\frac{b-a}{b}$ when 'tis decreaſing.

Wherefore, putting d=Difference between the two Terms a and b, the Logarithms of the ſame Ratio may be doubly expreſſed, and accordingly is either

$$\frac{1}{n}\times\overline{\frac{d}{a}-\frac{d^2}{2a^2}+\frac{d^3}{3a^3}-\frac{d^4}{4a^4},\ \&c.}\ \text{or}$$

$$\frac{1}{n}\times\overline{\frac{d}{b}+\frac{d}{2b^2}+\frac{d}{3b^3}+\frac{d}{4b^4},\ \&c.}$$

both producing the ſame Thing.

But if the Ratio of a to b be ſuppoſed to be divided into two Parts, *viz.* into the Ratio of a to the arithmetical Mean between the two Terms, and the Ratio of the ſaid arithmetical Mean to the other Term b, then will the Sum of the Logarithms of thoſe two Ratios be the Logarithm of the Ratio of a to b. Wherefore ſubſtituting $\frac{1}{2}s$ for $\frac{1}{2}a+\frac{1}{2}b$, and it will be, $\frac{1}{2}s : a :: 1 : 1-x$; whence $x=\frac{\frac{1}{2}s-a}{\frac{1}{2}s}=\left(\frac{s-2a}{s}=\right)\frac{d}{s}$; and again, as $\frac{1}{2}s : b :: 1 : 1+x$; $x=\frac{b-\frac{1}{2}s}{\frac{1}{2}s}=\left(\frac{2b-s}{s}=\right)\frac{d}{s}$: Therefore ſubſtituting $\frac{d}{s}$, for x, we ſhall have the Logarithms of thoſe Ratios; *viz.*

$$\frac{1}{n}\times\overline{\frac{d}{s}+\frac{d^2}{2s^2}+\frac{d^3}{3s^3}+\frac{d^4}{4s^4},\ \&c.}\ \text{and}$$

$$\frac{1}{n}\times\overline{\frac{d}{s}-\frac{d^2}{2s^2}+\frac{d^3}{3s^3}-\frac{d^4}{4s^4},\ \&c.}$$

The Sum of which two Logarithms, *viz.*

$$\frac{1}{n}\times 2\times\overline{\frac{d}{s}+\frac{d^3}{3s^3}+\frac{d^5}{5s^5}+\frac{d^7}{7s^7},\ \&c.}\ \text{is}$$

the Logarithm of the Ratio of a to b, whoſe Difference is d, and Sums s; which Series, without the Index n, is, by-the-bye, the Fluent of the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, aſſuming d, to be the flowing Quantity, for the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, is $\frac{2s\dot{d}}{ss-dd}=2\times\overline{\frac{\dot{d}}{s}+\frac{d^2\dot{d}}{s^3}+\frac{d^4\dot{d}}{s^5}+\frac{d^6\dot{d}}{s^7}+}$, &c. whoſe Fluent $2\times\overline{\frac{d}{s}+\frac{d^3}{3s^3}+\frac{d^5}{5s^5}+\frac{d^7}{7s^7}}$, &c. is *Neper's* Logarithm of $\frac{s+d}{s-d}$, and the ſame as above, abating the Index n. This Series, either Way obtained, converges twice as ſwift as the former, and conſequently is more proper for the Practice of making Logarithms: Thus put $a=1$, and b any Number at Pleaſure; then $\frac{d}{s}\mp\frac{b-1}{b+1}$, which aſſume $=e$, and then $b=\frac{1+e}{1-e}$; and becauſe $\frac{d}{s}=e$, therefore have we for

THEOREM I.

The Log. of $b=\left(\frac{1+e}{1-e}\mp\right)\frac{2}{n}\overline{\times e+\frac{1}{3}e^3+\frac{1}{5}e^5,\ \&c.}$

To illuſtrate this Theorem: Let it be required to find the Logarithm of 2 true to 7 Places.

Note, That the Index muſt be aſſumed of a Figure or two more than the intended Logarithm is to have.

EXAMPLE.

Here $(b=)\frac{1+e}{1-e}=2$; therefore $1+e=(\overline{1-e}\times 2=)$ $2-2e$; and $3e=(2-1=)\ 1$; whence $e=\frac{1}{3}$, and $ee=\frac{1}{9}$.

The OPERATION ſtands thus:

$$\begin{array}{rcr|rcr}
\frac{1}{3}=e & = & ,33333333 & e & = & ,33333333 \\
e^3 & = & 3703704 & \frac{1}{3}e^3 & = & 1234568 \\
e^5 & = & 411523 & \frac{1}{5}e^5 & = & 82305 \\
e^7 & = & 45725 & \frac{1}{7}e^7 & = & 6532 \\
e^9 & = & 5081 & \frac{1}{9}e^9 & = & 565 \\
e^{11} & = & 565 & \frac{1}{11}e^{11} & = & 59 \\
e^{13} & = & 63 & \frac{1}{13}e^{13} & = & 5 \\
\hline
 & & & & & ,34657359 \\
 & & & & & 2 \\
\end{array}$$

Whence *Neper*'s Logarithm of 2 is ,69314718

But ,69314718, multiplied by 3, will give 2,07944154 for the Logarithm of 8, inaſmuch as 8 is the Cube or third Power of 2; and the Logarithm of 8 + Log. of $1\frac{1}{4}$ is equal to the Logarithm of 10, becauſe $8 \times 1\frac{1}{4} = 10$; wherefore to find the Logarithm of $1\frac{1}{4}$ we have $b = \frac{1+e}{1-e} = 1\frac{1}{4} = \frac{5}{4}$; whence $e = \frac{1}{9}$, and $ee = \frac{1}{81}$.

The OPERATION ſtands thus:

$$\begin{array}{rcr|rcr}
\frac{1}{9}=e & = & ,11111111 & e & = & ,11111111 \\
e^3 & = & 137174 & \frac{1}{3}e^3 & = & 45725 \\
e^5 & = & 1693 & \frac{1}{5}e^5 & = & 339 \\
e^7 & = & 21 & \frac{1}{7}e^7 & = & 3 \\
\hline
 & & & & & ,11157178 \\
 & & & & & 2 \\
\end{array}$$

Whence *Neper*'s Logarithm of $1\frac{1}{4}$ is ,22314356
To which add the Logarithm of 8, 2,07944154
The Sum, *viz.* 2,30258510
is *Neper*'s Logarithm of 10. But if the Logarithm of 10 be made 1,000000, &c. as it is for Conveniency done in moſt of the Tables extant, then $\frac{2302585}{n} = 1{,}000$, &c. Whence $n = 2302585$, &c. is the Index for *Briggs*'s Scale of Logarithms; and, if the above Work had been carried on to Places ſufficient, the Index n would have been 2,30258, 50929, 94045, 68401, 79914,

79914, &c. and its Reciprocal, *viz.* $\frac{1}{n} = 0,43429,$ 44819, 03251, 82765, 11289, &c. which, by the Way, is the Subtangent of the Curve expreſſing *Briggs*'s Logarithms; from the Double of which the ſaid Logarithms may be had directly.

For, becauſe $\frac{1}{n} = 0,4342944$, &c. $\because \frac{2}{n} = ,8685889638$, &c. which put $= m$, and then the Logarithm of $b = \frac{1+e}{1-e} = me + \frac{me^3}{3} + \frac{me^5}{5} + \frac{me^7}{7} + \frac{me^9}{9}$, &c.

EXAMPLE.

Let it be required to find *Briggs*'s Logarithm of 2. Here $b = \frac{1+e}{1-e} = 2 \because e = \frac{1}{3}$, and $ee = \frac{1}{9}$.

The OPERATION ſtands thus:

m	=	,868588963			
me	=	,289529655	me	=	,289529655
me^3	=	32169962	$\frac{1}{3}\ me^3$	=	0723321
me^5	=	3574440	$\frac{1}{5}\ me^5$	=	764888
me^7	=	397160	$\frac{1}{7}\ me^7$	=	56738
me^9	=	44129	$\frac{1}{9}\ me^9$	=	4903
me^{14}	=	4903	$\frac{1}{11}\ me^{11}$	=	446
me^{13}	=	545	$\frac{1}{13}\ me^{13}$	=	42

Whence *Briggs*'s Logarithm of 2 is 0,301029993

AGAIN:

Let it be required to find *Briggs*'s Logarithm of 3. Now becauſe the Logarithm of 3 is equal to the Logarithm of 2 plus the Logarithm of $1\frac{1}{2}$ (for $2 \times 1\frac{1}{2} = 3$), therefore find the Logarithm of $1\frac{1}{2}$, and add it to the Logarithm of 2 already found, the Sum will be the Logarithm of 3, which is better than finding the Logarithm of 3 by the Theorem directly, inaſmuch as it will not converge ſo faſt as the Logarithm of $1\frac{1}{2}$; for the ſmaller the Fraction repreſented by e, which is deduced from the Number whoſe Logarithm is ſought, the ſwifter does the Series converge.

Here $b = \frac{1+e}{1-e} = \frac{3}{2} \because 2e + 2 = 3 - 3e \because e = \frac{1}{5}$, and $ee = \frac{1}{25}$.

The OPERATION is as follows:

m	$= ,868588963$			
me	$= ,173717792$		me	$= ,173717792$
me^3	$= 6948712$	$\frac{1}{3}$	me^3	$= 2316237$
me^5	$= 277948$	$\frac{1}{5}$	me^5	$= 55590$
me^7	$= 11118$	$\frac{1}{7}$	me^7	$= 1588$
me^9	$= 445$	$\frac{1}{9}$	me^9	$= 50$
me^{11}	$= 18$	$\frac{1}{11}$	me^{11}	$= 2$

Briggs's Logarithm of $1\frac{1}{2} =$,176091259
To which add the Logarithm of $2 =$,301029993
The Sum is the Logarithm of $3 =$ 0,477121252

Again, to find the Logarithm of 4, becauſe $2 \times 2 = 4$, therefore the Logarithm of 2 added to itſelf, or multiplied by 2, the Product 0,602059986 is the Logarithm of 4.

To find the Logarithm of 5, becauſe $\frac{10}{2} = 5$, therefore from the Logarithm of 10 1,000000000
ſubtract the Logarithm of 2 ,301029993
There remains the Logarithm of $5 =$,698970007

And becauſe $2 \times 3 = 6$; therefore

To find the Logarithm of 6,
To the Logarithm of 3 ,477121252
Add the Logarithm of 2 ,301029993
The Sum will be the Logarithm of $6 =$,778151245

Which being known, the Logarithm of 7, the next prime Number, may be eaſily found by the Theorem; for becauſe $6 \times \frac{7}{6} = 7$, therefore to the Logarithm of 6 add the Logarithm of $\frac{7}{6}$, and the Sum will be the Logarithm of 7.

EXAMPLE.

Here $b = \frac{1+e}{1-e} = \frac{7}{6} \because e = \frac{1}{13}$, and $ee = \frac{1}{169}$.

m =	,868588963		
me =	,066814536	me =	,066814535
me^3 =	395352	$\frac{1}{3}me^3$ =	131784
me^5 =	2339	$\frac{1}{5}me^5$ =	468
me^7 =	14	$\frac{1}{7}me^7$ =	2

Briggs's Logarithm of $\frac{7}{6}$	,066946789
To which add the Log. of 6	,778151245
The Sum is the Log. of 7 =	,845098034

Again, becauſe $4 \times 2 = 8$; therefore

To the Logarithm of 4	,60205998
Add the Logarithm of 2	,30102999
The Sum is the Logarithm of 8	,90308997

And becauſe $3 \times 3 = 9$; therefore

To the Logarithm of 3	,47712125
Add the Logarithm of 3	,47712125
The Sum is the Logarithm of 9 =	,95424250

And the Logarithm of 10 having been determined to be 1,00000000, we have therefore obtained the Logarithms of the firſt ten Numbers.

After the ſame manner the whole Table may be conſtructed; and as the prime Numbers increaſe, ſo fewer Terms of the Theorem are required to form their Logarithms; for in the common Tables, which extend but to ſeven Places, the firſt Term is ſufficient to produce the Logarithm of 101, which is compoſed of the Sum of the Logarithms of 100 and $\frac{101}{100}$, becauſe $100 \times \frac{100}{101} = 101$, in which Caſe $b = \frac{1+e}{1-e} = \frac{101}{100}$ $\because e = \frac{1}{201}$; whence, in making of Logarithms according to the preceding Method, it may be obſerved, that the Sum and Difference of the Numerator and Denominator of the Fraction whoſe Logarithm is ſought, is ever equal to the Numerator and Denominator of the Fraction repreſented by e; that is, the Sum of the Denominator; and the Difference, which is always Unity, is the Numerator; conſequently, the Logarithm of any prime Number may be readily had by the Theorem, having the Logarithm either next above or below given.

Tho' if the Logarithms next above and below that Prime are both given, then its Logarithm will be obtained ſomewhat eaſier. For half the Difference of the Ratios which conſtitutes the firſt Theorem, *viz.* $m\times\overline{\frac{dd}{2ss}+\frac{d^4}{4s^4}+\frac{d^6}{6s^6}+\frac{d^8}{8s^8}}$, *&c.* is the Logarithm of the Ratio of the arithmetical Mean to the geometrical Mean, which being added to the half Sum of the Logarithms, next above and below the Prime ſought, will give the Logarithm of that prime Number, which for Diſtinction-ſake, may be called *Theorem the ſecond*; and is of good Diſpatch, as will appear hereafter by an Example.

But the beſt for this Purpoſe is the following one, which is likewiſe derived from the ſame Ratios as Theorem the firſt. For the Difference of the Terms between ab and $\frac{1}{4}ss$, or $\frac{1}{4}aa+\frac{1}{2}ab+\frac{1}{4}bb$, is $\frac{1}{4}aa-\frac{1}{2}ab+\frac{1}{4}bb=\overline{\frac{1}{2}a-\frac{1}{2}b}^2=\frac{1}{4}dd=1$, and the Sum of the Terms ab and $\frac{1}{4}ss$ being put $=y$, therefore (ſince y in this Caſe $=s$, and $d=1$) it follows, that $\frac{1}{n}\times\overline{\frac{2}{y}+\frac{2}{3y^3}+\frac{2}{5y^5}+\frac{2}{7y^7}}$, *&c.* is the Logarithm of the Ratio of ab to $\frac{1}{4}ss$: Whence $\frac{1}{n}\times\overline{\frac{1}{y}+\frac{1}{3y^3}+\frac{1}{5y^5}+\frac{1}{7y^7}}$, *&c.* is the Logarithm of the Ratio of $\surd\, ab$ to $\frac{1}{2}s$, which converges exceeding quick, and is of excellent Uſe for finding the Logarithms of prime Numbers, having the Logarithms of the Numbers next above and below given, as in Theorem the ſecond.

EXAMPLE.

Let it be required to find the Logarithm of the prime Number 101; then $a=100$, and $b=102$; whence $y=20401$; put $\frac{1}{n}=m=,4342944819$, *&c.* Then the Series will ſtand thus, $\frac{m}{y}+\frac{m}{3y^3}+\frac{m}{5y^5}+\frac{m}{7y^7}$, *&c.*

And $m = ,43429$, *&c.* divided by $y = 20406$, quotes	,0000212879017.
Therefore to the half Sum of the Logarithms of 100 and 102 =	2,004300085881o
Add the ſaid Quote	0,0000212879017
And the Sum, *viz.*	2,0043213737827

is the Logarithm of 101 true to 12 Places of Figures, and obtained by the firſt Term of the Series only; whence it is eaſy to perceive what a vaſt Advantage the ſecond Term would have, were it put in Practice, ſince m is to be divided by 3 multiplied into the Cube of 20401.

This Theorem, which we'll call *Theorem the third*, was firſt found out by Dr. *Halley*, and a notable Inſtance of its Uſe given by him in the *Philoſophical Tranſactions* for making the Logarithm of 23 to 32 Places, by five Diviſions performed with ſmall Diviſors; which could not be obtained according to the Methods firſt made uſe of, without indefatigable Pains and Labour, if at all; on account of the great Difficulty that would attend the managing ſuch large Numbers.

Our Author's Series for this Purpoſe is (Page 357) $y \times \overline{\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}}$, *&c.* the Inveſtigation of which as he was pleaſed to conceal, induced me to inquire into it, as well to know the Truth of the Series, as to know whether this or that had the Advantage; becauſe Dr. *Halley* informs us, when his was firſt publiſhed, that it converged quicker than any Theorem then made public, and in all Probability does ſo ſtill. However that be, 'tis certain our Author's converges no faſter than the ſecond Theorem, as I found by the Inveſtigation thereof, which may be as follows:

From the foregoing Doctrine, the Difference of the Logarithms of $z-1$ and $z+1$ is $m \times \overline{\frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5} + \frac{2}{7z^7}}$, *&c.* which put equal to y, and the Logarithm of the Ratio of the Arithmetical Mean z, to the Geometrical Mean $\sqrt{zz-1}$ is $m \times \overline{\frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8}}$, *&c.* *per* Theo-

rem

near the second; for $z = \frac{1}{2}s$; whence $\frac{dd}{2ss} = \frac{1}{2zz}$.

Let A and B be the Logarithms of $z-1$ and $z+1$ reſpectively; then is $\frac{A+B}{2} + m \times \overline{\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}, \&c.}$ the Logarithm of z; and if the latter Part of the Series expreſſing the ſaid Logarithm of z be divided by the Series repreſenting the Difference of the Logarithms of $z-1$ and $z+1$, the Quotient will exhibit the Series required, viz. $\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.$ as appears by the following Operation:

$$\frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5}, \&c.\Big)\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}, \&c.\Big(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.$$

$$\underline{\frac{1}{2z^2} + \frac{1}{6z^4} + \frac{1}{10z^6}, \&c.}$$

$$\frac{1}{12z^4} + \frac{1}{15z^6}, \&c.$$

$$\underline{\frac{1}{12z^4} + \frac{1}{36z^6}, \&c.}$$

$$\frac{1}{180z^6}, \&c.$$

Now, becauſe the Dividend is ever equal to the Diviſor drawn into the Quotient of the Diviſion; it follows, that $y \times \overline{\frac{1}{4z} + \frac{1}{24z^3} + \frac{1}{360z^5}, \&c.}$ is equal to

$$m \times \overline{\frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6}, \&c.}$$

But $\frac{A+B}{2} + m \times \overline{\frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6}, \&c.}$ is the Logarithm of z; wherefore $\frac{A+B}{2} + y \times \overline{\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.}$ is the Logarithm of z. Q. E. I.

Note, I make the Author's 5th Term $\frac{13}{25200z^9}$, to be $\frac{1903}{226800z^9}$.

To illuſtrate this Theorem by an Example:

Let it be required to find the Logarithm of 101.

To the half Sum of the Logarithms of 100 and 102 = 2,0043000159

Add the Difference of the ſaid Logarithms divided by 4z equal to } 0,0000212876

And the Sum, *viz.* - - - 2,0043213735

is the Logarithm of 101 true to 9 Places of Figures: Whence it appears, that our Author's Series falls ſhort of Dr. *Halley*'s, in finding the Logarithm of the prime Number 101, three Places of Figures, by uſing only the firſt Terms of the Series; whereas, if two Terms in each were uſed, perhaps the Difference would have been conſiderably greater.

Note, This Series of our Author, deduced from Theorem the ſecond, is in Effect Dr. *Halley*'s too, but diſguis'd by being thrown into a different Form; which, however, has its Uſe, as being very ready in Practice.

Having thus inveſtigated ſeveral Theorems, whereby Tables of Logarithms, of any Form, may be conſtructed; it remains to ſhew how, from the Logarithm given, to find what Ratio it expreſſes.

The Logarithm of the Ratio of 1 to $1+x$ has been proved to be as $\overline{1+x}^{\frac{1}{n}}-1=\frac{1}{n}\times\overline{x-\frac{1}{2}x^2+\frac{3}{4}x^3-\frac{1}{4}x^4}$, &c. n being any infinite Index whatſoever; whence, if L be put for the ſaid Series, then $\overline{1+x}^{\frac{1}{n}}-1=L$; conſequently $\overline{1+x}^{\frac{1}{n}}=1+L$, and $1+x=\overline{1+L}^{n}=1+nL+\frac{1}{2}n^2L^2+\frac{1}{6}n^3L^3+\frac{1}{24}n^4L^4$, &c.

AGAIN:

The Logarithm of the Ratio of 1 to $1-x$ has likewiſe been proved to be as $1-\overline{1-x}^{\frac{1}{n}}=\frac{1}{n}\times x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4$, &c. $=L$; wherefore

$1 - \overline{1-x}^{\frac{1}{n}} = L$; and $\overline{1-x} = \overline{1-L}^{n} = 1 - nL + \frac{1}{2}n^2 L^2 - \frac{1}{6}n^3 L^3 + \frac{1}{24}n^4 L^4$, &c.

Whence $1 \pm x = 1 \pm nL + \frac{1}{2}n^2 L^2 \pm \frac{1}{6}n^3 L^3$, &c. is a general Theorem for finding the Number from the Logarithm given of any Species or Form whatſoever; but in the Application of it to Practice we labour under a great Inconvenience, eſpecially if the Numbers are large; that is to ſay, it converges ſo very ſlow, that it were much to be wiſhed it could be contracted.

However, if L be the Logarithm of the Ratio of *a* the leſſer Term, to *b* the greater, and either of them are given; then the other will be eaſily had, and expeditiouſly enough too:

For $\frac{b}{a}$ or $\frac{a}{b} = 1 \pm nL + \frac{1}{2}n^2 L^2 \pm \frac{1}{3}n^3 L^3$, &c.

Wherefore it follows, by the Help of a Table of Logarithms, that the correſponding Number to any Logarithm may be found, to as many Places of Figures as thoſe Logarithms conſiſt of: For, putting *d* equal to the Difference between the given Logarithm and the next leſs in the Table, then will the Number ſought, *viz.* $N = a \times \overline{1 + nd + \frac{1}{2}n^2 d^2 + \frac{1}{6}n^3 d^3}$, &c. But if *d* be put equal to the Difference between the given Logarithm, and the next greater, then $N = b \times \overline{1 - nd + \frac{1}{2}n^2 d^2 - \frac{1}{3}n^3 d^3}$, &c. both which Series converge faſter, as *d* is ſmaller.

But the firſt three Terms in each may be contracted into two, which is very uſeful, inaſmuch as it ſaves the Trouble of raiſing *n* and *d* in the third Term to the ſecond Power: For letting the firſt Term remain as it is, the other two are reduced to one, thus; make the ſecond Term the Numerator of a Fraction, and Unity minus the third Term divided by the ſecond is the Denominator.

Whence $N = a \times \overline{1 + nd + \frac{1}{2}n^2 d^2}$

becomes $N = a + \dfrac{ad}{\frac{1}{n}d - \frac{1}{2}d}$;

And $N = b \times \overline{1 - nd + \frac{1}{2}n^2 d^2}$

becomes $N = b - \dfrac{b \cdot d}{\frac{1}{n} + \frac{1}{2}d}$;

wherefore $a + \frac{ad}{m-\frac{1}{2}d}$, or $b - \frac{bd}{m+\frac{1}{2}d}$, will be the Number anſwering to the given Logarithm; which, tho' it differs a little from the Truth, is ſufficient to find the Numbers, exact to as many Places as *Briggs*'s Logarithms conſiſt of, *viz.* 14, which are the largeſt Tables extant. Much after the ſame Method may the whole Series be contracted; by which Means each alternate Power of d will be exterminated; or, which is the ſame Thing, every two Terms in the Series will be reduced to one, making the Whole ſhorter by Half.

To illuſtrate theſe Conſtructions by an Example:

Let it be required to find the Number anſwering to the Logarithm 7,5713740282 in *Briggs*'s Form.

From the given Logarithm - 7,5713740282
Subtract the Log. of 372710000 the next neareſt - - - - - - 7,5713710453
The Remainder is equal to $d=$,0000029829

And becauſe the Number 372710000 is leſs than the Number ſought, call it a, which, multiplied by ,0000029829, and the Product 1,111756659, &c. divided by $m-\frac{1}{2}d=$,4342929, &c. quotes 2559,92; which, added to 372710000, gives 372712559,92 for the Number ſought.

Thus, I preſume, the Doctrine of Logarithms has been ſufficiently exemplified, whether we conſider the Conſtruction of them for any given Numbers; or, on the contrary, the finding of the Numbers from the Logarithms given.

But, before I conclude, I ſhall give an Inſtance or two of the great Uſe of Logarithms in Arithmetical Calculations, and firſt in the purchaſing of Annuities.

If a be put for any Annuity, p for the preſent Value, r the Amount of One Pound for One Year at any Rate of Intereſt, and t for the Time or Number of Years the Annuity is to continue; then $p = \frac{a - \frac{a}{r^t}}{r-1}$, the Value of the Annuity.

EXAMPLE.

Let it be required to find the preſent Value of an Annuity of 60 *l. per Annum*, to continue 75 Years, at the Rate of 4 *per Cent. per Annum*.

Here $a=60$, $t=75$, and $r=1,04$. Now, in order to obtain the Anſwer, we muſt find the ſeventy-fifth Power of r, or of 1,04; that is, we muſt multiply 1,04 ſeventy-five Times into itſelf, which is exceeding tedious by the common Way, as any one may judge; but by the Logarithms 'tis done with the greateſt Eaſe; for if 0,0170333 the Logarithm of 1,04 be multiplied by 75, the Product 1,2774975 will be the Logarithm of the ſeventy-fifth Power of 1,04; which being ſubtracted from 1,7781512, the Logarithm of a equal to 60, will leave 0,5006537 the Logarithm of 3,167041, which being ſubtracted from 60, and the Remainder divided by $r-1=,04$ will give 1420,824 = 1420 *l.* 16 *s.* 5¾ *d.* for the Value of the Annuity; and if 1420,824 be divided by 60, the Quotient will exhibit the Number of Years Purchaſe requiſite to be given for any Annuity to continue 75 Years upon a good Security free of all Incumbrances, the Purchaſe being made at 4 *per Cent.*

Hence we ſee the Reaſon why the long Annuities purchaſed in the Year 1708, having about 75 Years to come, are valued in *Caſtain's* Bill of Exchanges at 24½ or 25 Years Purchaſe: For, tho' according to this Calculation, they are worth but a little more than 23 Years and a Half; yet, becauſe in the public Funds 4 *per Cent.* is ſcarcely ever made of Money, and the Contingencies it is there ſubject to, which thoſe Annuities, and other Government Securities, are not, makes them very juſtly worth 24½ or 25 Years Purchaſe.

Likewiſe Queſtions relating to Annuities upon Lives, whether for one, two, or three, &c. are almoſt as eaſily eſtimated. For Inſtance; it may readily be found by Logarithms, that an Annuity for a Man of Thirty, to continue during his Life, is worth 11,61 Years Purchaſe, Intereſt 6 *per Cent.* but at 4 *per Cent.* 14,68. And as the Probabilities of Life's Continuance, and the Value thereof, are determined by an algebraical Proceſs grounded upon the Rudiments of the Doctrine of Chances, and five Years Obſervations upon the Bills of Mortality of *Breſlaw*, the Capital of *Sileſia*; ſo there reſults that Truth and

Equity from the Operations, as ought to preſide in all Contracts of this Nature. Whence it follows, that all other Methods, whoſe Reſolution differs from this (eſpecially if the Difference be much), may juſtly be deemed erroneous, and conſequently prejudicial to one of the Parties concerned. Wherefore, to prevent Impoſitions thro' Ignorance, great Care ſhould be taken; which Precaution, however unneceſſary it may appear, 'tis preſumed, it will be regarded, inaſmuch as no one is willing to pay more Years Purchaſe than he has Chances for living; as, on the contrary, the Seller to receive leſs than his Due; which may poſſibly be by following the common Methods (where, for the moſt part, Regard is had neither to Age nor Intereſt) founded upon Caprice, Humour, or, if you pleaſe, Cuſtom, the Contract being made, as they can agree, right or wrong; which Method of Procedure ought to be exploded, ſince ſo liable to Error, and the Conſequences drawn therefrom ſo often wide of the Truth.

The other Inſtance which I ſhall give of the great Uſe of Logarithms is in the Caſe of *Seſſa*, as related by Dr. *Wallis* in his *Opus Arithmeticum* from *Alſephad* (an *Arabian* Writer), in his Commentaries upon *Tograius*'s Verſes; namely, that one *Seſſa*, an *Indian*, having firſt found out the Game at *Cheſſe*, and ſhewed it to his Prince *Shebram*, the King, who was highly pleaſed with it, bid him aſk what he would for the Reward of his Invention; whereupon he aſked, That for the firſt little Square of the *Cheſſe* Board, he might have one Grain of Wheat given, for the ſecond two, and ſo on, doubling continually, according to the Number of the Squares in the *Cheſſe* Board, which was 64. And when the King, who intended to give a very noble Reward, was much diſpleaſed, that he had aſked ſo trifling an one, *Seſſa* declared, that he would be contented with this ſmall one. So the Reward he had fixed upon was ordered to be given him. But the King was quickly aſtoniſh'd, when he found, that this would riſe to ſo vaſt a Quantity, that the whole Earth itſelf could not furniſh out ſo much Wheat. But how great the Number of theſe Grains is, may be found by doubling one continually 63 Times, ſo that we may get the Number that comes in the laſt Place, and then one Time more to have the Sum of all; for the Double of the laſt Term leſs by one is the Sum of all. Now this will be moſt expeditiouſly done by Logarithms, and accurately enough too for this Purpoſe: For if to the Logarithm of 1, which is 0, we add the Logarithm of 2 (which is 0,3010330) multiplied by 64, that is, 19,2659200, the abſolute Number agreeing to this will be greater than 18446, 00000, 00000, 00000, and leſs than 18447, 00000, 00000, 00000.

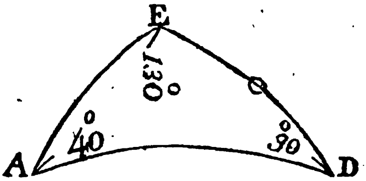
As I have had the reviſing of theſe Sheets, ſo it may be expected that I ſhould give my Opinion concerning Mr. *Cunn* and our Author, in regard to Spherical Trigonometry; wherein the former accuſes the latter, and ſeveral other eminent Authors, of having committed many Faults, and, in ſome Caſes, of being miſtaken, eſpecially in the Solution of the 12th Caſe of Oblique Spherics; in which Mr. *Cunn* has intirely miſtaken the Author's Meaning, as plainly appears by his Remark, where he conſtitutes a Triangle whoſe Sides are equal to the given Angles; whereas the Author means, that each Angle ſhould firſt be changed into its Supplement, and then with the ſaid Supplements another Triangle conſtituted, whoſe Angles, by the very Text of the 14th Propoſition of his own Spherical Trigonometry, will be the Supplements of the Sides ſought in the given Triangle; to which Propoſition I refer the Reader. That this is the Senſe of the Author, is very evident, if impartially attended to, and which I think could poſſibly have no other Meaning; and accordingly aver what is here advanced to be univerſally true; but, becauſe I would not be miſunderſtood, ſhall illuſtrate the Truth thereof by a numerical Operation; which, to thoſe who care not to trouble themſelves with the Demonſtration, may be ſufficient; and, to others, ſome Satisfaction.

EXAMPLE.

Suppoſe, in the Oblique-angled Spherical Triangle A D E, there are given the Angles A, D, E, as *per* Figure, and the Side DE required.

Note, Write down the Supplements of the two Angles next the Side required firſt; and then the Operation may ſtand thus;

The



The Supplement of the Angle.	E = 50°	Sine Co. Ar. -	0,115746
	D = 150	Sine Co. Ar. -	0,301030
	A = 140	Sine 220° - -	9,937530
	Sum = 340	Sine 20 - -	9,534052
	2)		
	½ Sum = 170		19,888358
			2)
½ Sum, minus the Supplement of the Angle.	E = 120		
	D = 20	- - - - -	9,944179

Which laſt Figures 9,944179 give the Side of 61° 34′; and the Double thereof, *viz.* 123° 08′; ſubtracted from 180 Degrees, leaves for the Supplement 56° 52′, which is the Side of D E required.

The Rule which Mr. *Cunn* ſubſtitutes in the Room of our Author's, is alſo univerſal (but not new); and, conſequently, when he ſays, Change one of the Angles adjacent to the Side ſought into its Supplement, it is very juſt; though, by the way, I affirm, it is equally true, if the Angle oppoſite to the Side ſought were changed into its Supplement (which perhaps is what has not yet been taken Notice of); only then, inſtead of having the Side ſought directly, we ſhould have its Complement to 180 Degrees, as in the preceding Example; but there is a Neceſſity of changing either one or all the Angles into their Supplements, though it is beſt to change only one, which let be either of thoſe next the Side ſought, no matter which; and the Side will be had directly without any Subduction, as will appear by the ſubſequent Operation.

EXAMPLE.

Let the Angle E be changed into its Supplement, and the Side D E ſought; which Supplement, and the other Angle adjacent to the Side ſought, being written down firſt, the Operation may be as follows:

Sup. of the Angle $E=50$ Sine Co. Ar.—0,115746
The $\left\{\begin{array}{l} D=30 \\ A=40 \end{array}\right.$ Angle Sine Co. Ar.—0,301030
Sine 10° - 9,239670
Sum 120 Sine 30 - 9,698970
2|——
½ Sum 60 Sum 19,355416

½ Sum minus $\left\{\begin{array}{l} \text{Sup. Angle } E = 10 \\ \text{Angle } D = 30 \end{array}\right.$ ½ Sum 9,677708

Which half Sum 9,677708 gives the Sine of 28° 26′, and the Double thereof 56° 52′ is the Side D E ſought, the ſame as before, when all the Angles were changed into their Supplements.

Whence it is abundantly manifeſt, that thoſe two Methods of Operation, notwithſtanding their Manner is ſo different, agree preciſely in Practice; and, conſequently, we may conclude our Author's Rule to be right. Wherefore I wonder Mr. *Cunn* did not attend better to the Words of our Author's Rule, before he ventured to attack the Characters of ſo many famous Trigonometrical Writers. But to remove the Imputation of the Charge againſt thoſe Authors who have deſerved ſo well of the Mathematics, and to juſtify them to the World (for Juſtice ought to have Place), it is, that I have ventured to give my Opinion, and point out where Mr. *Cunn* was miſtaken: The Reaſon of which is not eaſily aſſigned, ſince, to give him his Due, it could not be for want of Knowledge, tho', in this Caſe, I can't think it intirely owing to Inadvertence, inaſmuch as it was a premeditated Thing; and I am loth to impute it to any contentious Inclinations of his, in diſputing the Veracity of our Author's Rule, becauſe it did not appear with all that Plainneſs requiſite to prevent carping by the Litigious: Wherefore, as I am in Suſpenſe how to determine, I ſhall leave the Deciſion thereof to better Judgments.

Indeed, Mr. *Heynes*'s Rule, which directs with the three Angles given to project a Triangle, as if they were Sides, is deficient, were it only on that very Account: For with the given Angles, in the preceding

Example, it will be impoſſible to conſtruct a Triangle; becauſe 'tis requiſite, that two Sides together, however taken, be greater than the third; whereas, in this Caſe, they will be leſs: But the Rule is not only deficient in that Reſpect, but really wrong: For tho' what Mr. *Heynes* aſſerts is juſt, *viz.* that the greateſt Side in the ſupplemental Triangle is the Supplement of the greateſt Angle in the other Triangle; yet, notwithſtanding that, the Conſequence drawn therefrom is falſe, and ſo the Solution only imaginary: For, with Submiſſion, neither the Sides, nor their Supplements, in Mr. *Heynes*'s ſupplemental Triangle, are the Meaſures of the Sides ſought. 'Tis true, when one of the Angles is a Right one, and the others both acute, then the ſaid ſupplemental Triangle is that wanted to be conſtructed, as containing all the given Angles; and, conſequently, the Sides appertaining thereto are the very Sides required: But then this is only one Inſtance out of the infinite Number of other Triangles that may be conſtructed, and which is not ſolved directly by the Triangle firſt projected neither; for the greateſt Angle thereof muſt be changed into its Supplement, when the Side oppoſite to the Right Angle is requir'd; and if the Right Angle ſtill remains, and either one or both of the other given Angles are obtuſe, the Solution is render'd more perplex'd: Wherefore there can be no general Solution given to any Triangle, by conſtituting a Triangle whoſe Sides are equal to the given Angles, except to that particular one which Mr. *Cunn* takes Notice of in his Remark, where each given Angle is the Meaſure of its oppoſite Side ſought, and which therefore needs no Operation.

This I thought myſelf obliged to obſerve, in Juſtice to Mr. *Cunn*, who, we ſee, is not intirely to blame; as having juſt Reaſon to object againſt the Veracity of Mr. *Heynes*'s Rule, tho' not againſt the Rules of the other Authors by him nominated.

And here I can't but take Notice of ſome Gentlemen, who are ſo very fond of finding Fault, that, rather than you ſhall not be in the Wrong, they will wreſt your own Meaning from you, and will not ſuffer an Error, tho' ever ſo minute, to paſs, without proclaiming it to the Public, under Pretence of preventing their being impos'd upon; whereas, if the Truth were known, I fear it would appear to be the Vanity of their Hearts, an Over-fondneſs of being thought wiſer and more knowing than the reſt of Mankind; nay, I think, it appears plainly ſo, by their oppoſing the Works of Men greater than themſelves: But if, inſtead of comparing how far their finite Knowledge extend, or exceeded another Perſon's, they conſider'd how much there was they knew nothing of; as it would conduce to make them humble, ſo, I am of Opinion, it would contribute very much toward their leaving off that Manner of Writing. Beſides, as I take it, the Buſineſs of Writing is not ſo much to diſcover who has committed the moſt Faults, as to avoid them, and make greater Improvements.

But, what is the moſt to be wonder'd at, thoſe who are ſo very ready in finding Fault, not without great Suſpicion, receive the beſt Part of their Knowledge from the Works of thoſe very Authors againſt whom they exclaim. The Reaſon that induces me to think ſo is this: Whilſt they are ſtudying an Author, in order to underſtand him, then it is, perhaps, they diſcover ſomething which he was pleaſed to omit, or thought fit to conceal, for which 'tis more than probable they take Care not to omit paying a profound Reſpect to their vainly-imagined ſuperior Geniuſes: And if, by Accident, an Error ſhould creep in (which is very poſſible, none being infallible), then, to be ſure, he muſt be egregiouſly miſtaken, and not underſtand what he was about: But, I ſay, this Diſquiſition into the Demerits of an Author would never have been made, had they underſtood the Subject beforehand; for, if otherwiſe, they muſt be of a ſad Cynical Temper, as well as have little elſe to do, to make it their Buſineſs to diſcover Faults, and at the ſame Time acknowledge not one ſingle Beauty; a very ungrateful Return for the Advantage they receive in the Peruſal.

Nor do they do the Public that Service they pretend to: For thoſe that are capable, and will be at the Trouble, of reading a Treatiſe upon a Subject without a Maſter, are as well able as themſelves to rectify what is amiſs; and as for thoſe who will not be at that Trouble, there is no Danger of their being led aſtray; ſince it is the ſame Thing to them, whether there be any Miſtakes, or not.

However, if, after all, there ſhould be a Neceſſity for an Admonition, why can't it be done with Candour and

Humanity? And then, without doubt, an Author, out of Regard to Truth, which of all Things ought to be preferred, would be thankful: And to reprove otherwiſe, is to be ungenerous; becauſe, whenever thoſe Miſtakes happen, as they are for the moſt part owing more to Inadvertency, than Want of Knowledge; ſo they ſhould therefore be attributed to the Frailty of human Nature (to which we are all more or leſs ſubject), nothing being more common amongſt all Profeſſions, than the writing of one Thing for another.

If any think, by my interfering between our Author and Mr. *Cunn*, that I have run into the ſame Error, of which I accuſe others in general of being guilty, let them pleaſe to conſider that I have only writ in the Vindication of Gentlemen, who were firſt wrongfully accus'd, and in one Particular juſtify'd Mr. *Cunn:* For ſuch an Occaſion as this offering, I thought the Difference between them lay upon me to decide, leſt I ſhould be taxed with Partiality for not doing Juſtice, or with Ignorance in not determining an Affair which held ſome in Suſpenſe to know who was in the right or wrong; for there could be no Poſſibility of making a Merit in adjuſting a Thing of ſo eaſy a Nature; tho', perhaps, to conceive thoroughly the Reaſon of all the different Methods of Solution, may not be ſo eaſy neither.

But, to proceed: As for the Omiſſions our Author has made in not determining accurately when ſome of the Caſes are ambiguous, and when not, I ſhall not quarrel with thoſe who think him to blame; but, if I may be allowed to give my Opinion, I think they are determin'd for the moſt part, as well, or, at leaſt, with more Eaſe, from the Conſtruction of the Triangles, becauſe it fixes an Idea of what one is about, by exhibiting a kind of an ocular Demonſtration; and, conſequently, prevents the laying of that Streſs upon the Memory, as all thoſe are obliged to who depend intirely upon Mr. *Cunn*'s Rules, which to Beginners is not very agreeable: Hence, who knows but that what our Author wrote relating to the ambiguous Caſes, he thought ſufficient? That is, that the Reader would not ſtop, for want of farther Explications, but with more Eaſe ſupply himſelf with what was wanting when he came to the Practice thereof, I mean the Conſtruction of Triangles (for, after all, without the Knowledge of that, a Perſon will have but a mean Notion of this uſeful Branch of the Mathematics); and, if ſo, he ought in ſome meaſure to be excuſed, eſpecially if to this we join the following Conſideration, *viz.* that few or none ever learn Spherical Trigonometry, purely for the Sake of calculating Sides and Angles, to determine their Ambiguities; beſides, what is ambiguous in Trigonometry, is very often not ſo in Geography and Aſtronomy, *&c.* for which the other is chiefly learnt.

For Inſtance: If we know the Latitude of *London*, and the Diſtance and Difference of Longitude between the ſaid Place and *Rome*, notwithſtanding there are two Sides, and the Angle oppoſite to one of them, given, the Caſe is not doubtful when we undertake to find the Latitude of *Rome*; unleſs it be not known whether it lies to the *Northward* or *Southward* of *London*; which however could not be determined by any Principles of Trigonometry. Likewiſe, in Aſtronomy, if the Latitude of the Place, the Sun's Declination and Azimuth, were given, the *Quæſitum* is not doubtful neither, unleſs the Sun's Declination exceeds the Latitude of the Place, and both are of the ſame Denomination, that is, both *North* or both *South*; in which Caſes, becauſe it is poſſible for the Sun to be upon the ſame Azimuth Circle, twice in the Forenoon, and upon another Azimuth Circle, twice in the Afternoon; it is doubtful, if by Circumſtances, during the Obſervation, we can't diſcover which of the Times, whether the firſt, or laſt; but if thoſe Times fall near each other, it will be quite impoſſible to diſtinguiſh which, and therefore ambiguous. Other Inſtances might be produced, but I believe theſe are ſufficient to evince, that thoſe nice Diſtinctions are not ſo neceſſary in Practice: If there be thoſe who think otherwiſe, I ſhall not diſpute it, but leave them to their Opinion without Interruption.

However, what with Mr. *Cunn's* Rules for determining the ambiguous Caſes (which are judiciouſly drawn up, as including all the Varieties poſſible), and the Corrections now made by reſtoring what was loſt and corrupted, our Author's Treatiſe of Trigonometry, in reſpect to Theory, may perhaps appear compleat, even to the moſt ſcrupulous. And.

Here I thought to conclude; but, for the Sake of Novelty, and to illuſtrate the various Methods for ſolving

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ing the 12*th* Caſe of Oblique Spherics, where the three Angles are given to find either of the Sides, I ſhall beg Leave to give one Inſtance more, in order to ſhew how it may be perform'd after a new Manner, by the Help of the natural and logarithmetic Verſed Sines; which, if not intirely new, is not ſo publicly known as the preceding Methods; at leaſt, I never ſaw any-where the Method of Operation, and therefore ſhall deliver a Rule for that Purpoſe, in the following Words:

RULE.

Having according to the former Directions, chang'd one of the Angles next the Side ſought into its Supplement; take the natural Verſed Sine of the Difference of the ſaid Supplement and the other adjacent Angle, and ſubtract it from the natural Verſed Sine of the Angle oppoſite to the Side ſought, and to the Logarithm of the Remainder add the Square of the Radius; then from the Sum ſubtract the logarithmetic Sines of the above Supplement, and the ſame adjacent Angle; and the Remainder is the Logarithm of a Number, which will be the Verſed Sine of the Side ſought.

EXAMPLE.

Supplement < E = 50°
Angle D = 30
Natural V. Sine { Diff. = 20 = ,06030
Natural V. Sine { < A = 40 = ,23395
,17365
The Log. of which Diff. ,17365 with the Square of Radius, is } 29,239674
Sine of the Sup. of the < E 50° = 9,884254
Add the Sine of the < D 30 = 9,698970
Sum ſubtract 19,583224
Remains 9,656450

Which Remainder 9,656450 gives the Logarithm Verſed Sine of D E 56° 42', agreeing exactly with the former Computations.

Note, If the ſaid Remainder exceeds 10,000000, it implies that the Side ſought is greater than a Quadrant wherefore cancelling the Characteriſtic 10, look out for the Number anſwering the remaining Logarithm from which cut off the Left-hand Figure, or, which is the ſame Thing, abate the Radius (*viz.* Unity); and the Remainder will be the natural Sine of the Exceſs of the Side ſought above a Quadrant.

As the natural and logarithmetic Verſed Sines are not ſo frequently met with in Books as the artificial Sines, 'tis poſſible, on that Account, this Rule may meet with ſome Objection; for which Reaſon, and not knowing whether it may be thought preferable to the foregoing Methods (tho' undoubtedly very eaſy in Practice), I have omitted its Demonſtration; but have publiſhed the Rule, with ſome View of introducing the Uſe of the former Sines, which ſometimes are preferable to the latter: For by the Help of the ſaid Verſed Sines, and the Reaſoning uſed in obtaining this Rule, we neceſſarily come to the Knowledge of ſolving that Problem, where two Sides and the contained Angle are given, and the third Side required, at one Operation, very uſeful in Aſtronomy and Geography, eſpecially in the latter; when the Latitudes and Longitudes of two Places are given to find their Diſtance aſunder: But the Rule for performing it, and the Demonſtration thereof, is alſo omitted for the Sake of Brevity.

However, 'tis eaſy to perceive, ſince Angles may be turned into Sides, that the preſent Rule includes the Solution of that uſeful Problem in Aſtronomy for finding the Sun's Azimuth, having the Latitude of the Place, the Sun's Altitude and Diſtance from the elevated Pole given; by which means the Variation of the Compaſs, of ſuch Importance to Navigators, may be readily determined in any Part of the World.

An Example of which, comprehending the latter Part of the Rule (*viz.* when the Remainder exceeds 10,000000) is exhibited.

EXAMPLE.

Suppoſe on *June* the 30th, 1732, at *London*, in the Latitude of 51° 52′ N. it were required to find the Sun's true Azimuth, when his Altitude was 50° 00′, in the Afternoon. Firſt,

From

		° ′	
From the Com. of the Altitude	-	40 00	
Sub. the Com. of the Latitude	-	38 28	
Natural V. Sine of the Difference	- -	1 32	,00035
Natural V. Sine of the Sun's Diſt. from the Pole		67 54	,62377
			,62342
The Log. of which Difference ,62342 with the Square of the Radius, is			29,794780
Coſine of the Latitude 51° 32′	- -		9,793831
Coſine of the Altitude 50 00	- -		9,808097
		Sum ſubtract	19,601898
		Remains	10,192882

Here the Remainder exceeds 10,000000; wherefore cancel the Characteriſtic 10, and the Number anſwering the remaining Logarithm is 1,5591; the Exceſs of which above Unity, *viz.* ,5591, gives the natural Sine of 34° 00′; whence the Sun's true Azimuth is North 124° 00′ Weſt: At which Time, if the Sun's Magnetical Azimuth were North 110° 30′ Weſt, the Variation of the Compaſs would be 13° 30′ Weſt, as appears by the following Subtraction.

True Azimuth North,	124°	00′	Weſt
Mag. Azimuth North,	110	30	Weſt
Variation	13	30	Weſt

N. B. If the Sun's Declination had been South, then the Verſed Sine of the Sun's Diſtance from the elevated Pole would have been equal to Unity plus the natural Sine of the Sun's Declination; which in Practice creates no more Trouble than when the Declination is North, if ſo much; ſince it is at leaſt as eaſy to take the natural Sine of an Arc, as to take the Verſed Sine of its Complement to 90 Degrees; which Sines, and others, with their reſpective Logarithms, *&c.* may readily be had out of *Sherwin's* Mathematical Tables.

FINIS.

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